The Open University

MS221
Exploring Mathematics

Chapter A3

Functions from geometry

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Functions from geometry

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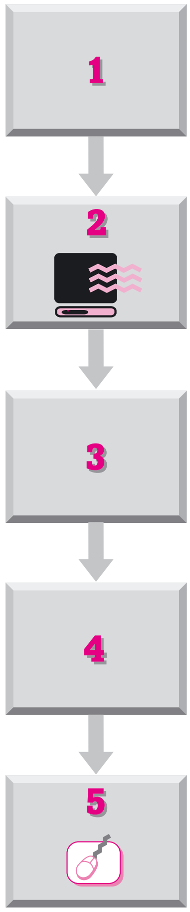
Study guide

There are five sections to this chapter. They are intended to be studied consecutively in five study sessions. Section 2 requires the use of a DVD player and Section 5 requires the use of the computer together with Computer Book A.



The pattern of study for each session might be as follows.

Study session 1: Section 1.

Study session 2: Section 2.

Study session 3: Section 3.

Study session 4: Section 4.

Study session 5: Section 5.

Each session requires two to three hours, the longest being the second and the fourth.

Before studying this chapter, you should be familiar with the following topics:

- the notations (a, b), $[a, b]$, and so on, for open and closed intervals of $\mathbb{R}$;
- the concept of a real function;
- basic properties of the trigonometric functions sine, cosine, tangent, cosecant, secant and cotangent, and the function arctan;
- exact values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\theta = \frac{1}{6}\pi$, $\frac{1}{4}\pi$, $\frac{1}{3}\pi$ and related angles;
- scaling and translation of the graphs of real functions.

The optional Video Band A(iv) *Algebra workout – Trigonometry* could be viewed at any stage during your study of this chapter.

Introduction

The mathematical concept of a *function*, which acts as a processor converting inputs to outputs, is remarkably versatile. In this chapter we concentrate on functions that arise from geometry. This means that the inputs and outputs of the functions we consider consist of points from the Cartesian plane or from the real line.

Broadly speaking, geometry is about the properties of figures in the plane or in space. For example, given two ellipses in a plane a typical geometrical question is to ask whether the two ellipses are congruent – that is, have the same size and shape. A simple thought experiment suggests a way to answer this question. Just pick up one of the ellipses and attempt to superimpose it on top of the other ellipse. If the two ellipses coincide exactly, then we can conclude that they do indeed have the same size and shape.

This experiment raises several questions. What is the best way to describe a figure (such as an ellipse) so that we can manipulate it mathematically? How do we represent the movement of a figure from one position in the plane to another? These and many other questions can be answered using the language of functions. In this chapter the use of this language is illustrated by continuing the investigation of the shape of quadratic curves begun in Chapter A2.

In Section 1, the general definition of a function is given, and illustrated with several geometrical examples. You will see ways of representing these functions graphically.

Section 2 introduces functions called *isometries* that can be used to move figures in the plane, by translating them, rotating them about the origin, or reflecting them in a line through the origin. You will see that two or more of these basic functions can be applied in succession to produce more general *composite* functions.

In Section 3, rotations are used to derive trigonometric sum formulas. These are then used to develop further trigonometric formulas and techniques that will be needed in Section 4.

Section 4 continues the investigation of quadratic curves started in Chapter A2. In particular, a strategy is developed for sketching any quadratic curve with equation of the form

$$Ax^2 + Bxy + Cy^2 + F = 0 \quad (\text{where } B \neq 0).$$

Section 5 contains all the computer work for the chapter. You will see how to use the computer to plot triangles and conics, before and after they have been moved by various isometries.

1 Functions

1.1 What is a function?

The concept of a *function* provides a convenient language with which to discuss a wide variety of situations in which a mathematical process is applied to given inputs in order to produce required outputs. For example, the expression

$$f(x) = \sqrt{x} \quad (x \geq 0) \tag{1.1}$$

defines a function called f. This function f takes non-negative real numbers x as inputs and applies the process of taking the square root of x to produce non-negative real numbers $\sqrt{x}$ as outputs. In particular, on inputting the number 4, we obtain the output $\sqrt{4} = 2$. Such a function f, whose inputs and outputs are all real numbers, is called a **real function**.

Real functions were introduced in MST121, Chapter A3, Section 1.

Real functions are ideal for manipulating numerical data but in other contexts, such as geometry, the inputs and outputs may not be real numbers. For example, they may be points in the plane, or even sets in the plane (such as conics). Therefore, a general concept of 'function' is introduced; it is illustrated as a 'processor' in Figure 1.1.

Other words for function are *mapping* and, in geometric contexts, *transformation*.

A **function** f is defined by specifying:

(a) a set X, called the **domain** of f;

(b) a set Y, called the **codomain** of f;

(c) a **rule**, or process, that associates with each x in X a unique y in Y; we write $y = f(x)$ and call $f(x)$ the **image** of x under f.

To indicate that the sets X and Y are general, they are shown here as ovals. Individual elements (or members) of sets are shown as points.

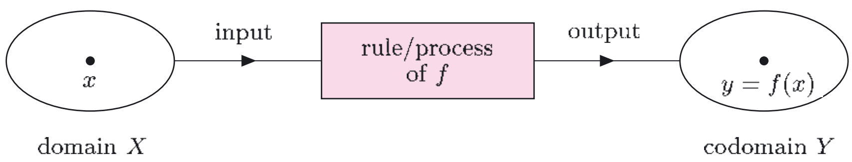


Figure 1.1 A function f as a processor

When specifying such a general function, the following *two-line notation*, that summarises the content of Figure 1.1, is often used.

This notation is read as: 'the function f from X to Y which maps x to $f(x)$'. Notice the different types of arrows used.

$$\begin{aligned} f : X &\longrightarrow Y \\ x &\longmapsto f(x) \end{aligned}$$

To illustrate this notation, consider the real function f defined in equation (1.1). Here the set of input values is the interval $[0, \infty)$, so this set is the domain. The codomain can also be taken to be $[0, \infty)$, since all output values lie in $[0, \infty)$, or any larger set such as $\mathbb{R}$ itself. For simplicity, we usually take the codomain of a real function to be $\mathbb{R}$.

The rule of f can be expressed either as $f(x) = \sqrt{x}$ or as $x \longmapsto \sqrt{x}$. So we can specify f using the notation

$$\begin{aligned} f : [0, \infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sqrt{x}. \end{aligned}$$

This two-line notation appears somewhat cumbersome compared to equation (1.1), for which it has to be assumed that the codomain is $\mathbb{R}$. For more general functions it is helpful to use the two-line notation, which makes clear the domain, codomain and rule of the function. To indicate only the domain and codomain of the function, we simply write

$$f : X \longrightarrow Y.$$

The word 'function' was coined by the German mathematician and philosopher Gottfried Wilhelm Leibniz, and the notation $f(x)$ was first used by the Swiss mathematician Leonhard Euler. The two arrow notations for functions were introduced in the 20th century, with $f : X \longrightarrow Y$ first appearing around 1940 and the barred arrow $x \longmapsto f(x)$ in the early 1960s.

Both Leibniz (1646–1716) and Euler (1707–1783) made major contributions in many areas of mathematics. Leibniz invented calculus, at about the same time as Newton, and Euler was the most prolific mathematician of all time.

In this section, we consider two types of function arising from geometry. In each case, the domain and codomain are subsets of $\mathbb{R}$ or of the Cartesian plane. The standard notation for this plane is $\mathbb{R}^2$, reflecting the fact that each point in the plane can be represented as an ordered pair of real coordinates; see Figure 1.2(b).

Pronounce $\mathbb{R}^2$, as 'R two'.

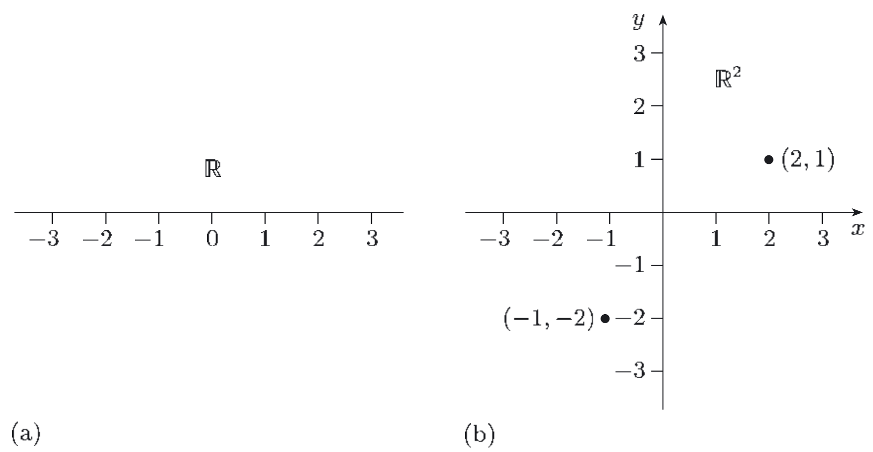


Figure 1.2 Line $\mathbb{R}$ and plane $\mathbb{R}^2$

1.2 Two geometric functions

Each of the functions introduced in this subsection is related to the quadratic curve with equation

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0.$$

See Chapter A2, Section 4, Frame 1.

This curve is an ellipse, which for convenience will be called E throughout this subsection.

A translation function

In the previous chapter a method was developed for deciding if a quadratic curve such as E is an ellipse, a parabola, or a hyperbola. Since the equation of E can be rearranged, by completing the squares, in the form

$$\tfrac{1}{9}(x-2)^2 + \tfrac{1}{4}(y-1)^2 = 1, \tag{1.2}$$

This form of the equation of E is equivalent to

$$\frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} = 1,$$

used in Chapter A2.

it follows that E is an ellipse with centre at the point $(2, 1)$. For if (x, y) is any point on E, then (x, y) must satisfy equation (1.2), so the point $(x - 2, y - 1)$ lies on the ellipse in standard position with equation

$$\tfrac{1}{9}x^2 + \tfrac{1}{4}y^2 = 1.$$

In other words, we can move from any point (x, y) on E to the point $(x-2, y-1)$ on the ellipse in standard position by translating two units to the left and one unit down; see Figure 1.3, which shows both ellipses.

In Chapter A2, Section 4, Frame 1, the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ was translated onto E.

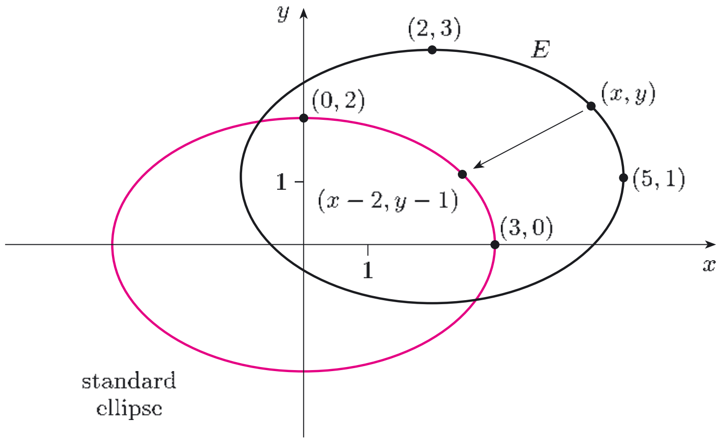


Figure 1.3 Translating E onto the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$

The following activity will help to clarify what is happening here.

Activity 1.1 Translating a point

Use equation (1.2) to check that the point $(5, 1)$ lies on E. Write down the point obtained by translating $(5, 1)$ two units to the left and one unit down, and check that the translated point lies on the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$.

A solution is given on page 48.

Figure 1.3 shows the effect that the translation has on the ellipse E. But the same translation could, if we wished, be applied to any point (or set of points) of the plane $\mathbb{R}^2$. So let us forget about the ellipse for the moment and concentrate on how a function can be used to translate arbitrary points of the plane two units to the left and one unit down.

Suppose that a processor calculates this translation. The processor has to accept any point (x, y) from $\mathbb{R}^2$, calculate the point $(x-2, y-1)$ and output the result back to $\mathbb{R}^2$. This suggests that we think of the translation as a function, t say, with domain and codomain $\mathbb{R}^2$. This function is defined in two-line notation as follows:

Here the function name t is chosen for the translation. Using an appropriate name is not essential.

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x-2, y-1). \end{aligned} \tag{1.3}$$

For example, the image under t of the point $(4, 5)$ is the point $t(4, 5) = (4-2, 5-1) = (2, 4)$. A function that describes a translation is called a **translation function**. For simplicity, such a function is often referred to as a 'translation'.

Strictly speaking we should write $t((4, 5))$ here, but the outer brackets are usually omitted to reduce clutter.

Activity 1.2 Using a translation function

Determine the image of each of the points $(1, -3)$, $(2, 7)$ and $(-2, 4)$ under the above translation function.

A solution is given on page 48.

It is important to realise that the rule $(x, y) \longmapsto (x-2, y-1)$ describes the effect of the translation t on each *point* (x, y) of the domain $\mathbb{R}^2$. Often, however, we are interested in the effect that a function has on a whole *subset* of points in the domain. For example, the conic E is a subset of the domain $\mathbb{R}^2$ and we know that t translates this subset onto the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$.

If each point of a set A is also a point of a set X, then A is called a **subset** of X. This definition allows X to be thought of as a subset of itself.

In general, let $f : X \longrightarrow Y$ be a function and A be a subset of X, as in Figure 1.4. Then the set of all images $f(x)$ with x in A, is called the **image of A under f**, written $f(A)$.

Notice that $f(x)$ and $f(A)$ are two quite different uses of notation. The first refers to the image of a *point* of the domain, whereas the second refers to the image of a *subset* of the domain.

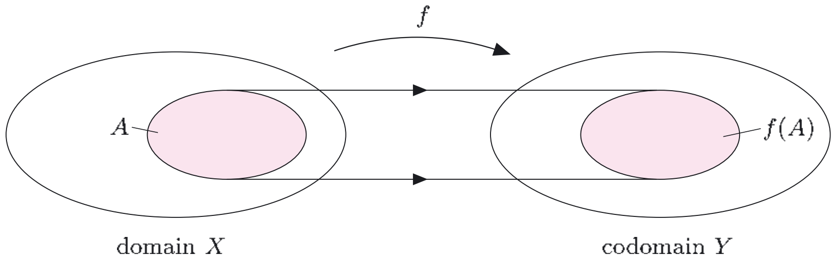


Figure 1.4 The image $f(A)$

If the subset A is the domain X itself, then $f(X)$ is called the **image set** of f; it is a subset of the codomain of f.

In general, the image set may or may not be the whole codomain.

For example, if t is the translation function specified in equation (1.3) and E is the ellipse with equation (1.2), then $t(E)$, the image of E under t, is the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$. The domain of t is $\mathbb{R}^2$ and the image set of t is $t(\mathbb{R}^2) = \mathbb{R}^2$.

Activity 1.3 Finding a translation function

Define a translation function t that maps the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ onto E, giving your answer in two-line notation.

A solution is given on page 48.

A parametrisation function

Next we consider a function that can be used to plot points of E. It is based on the idea of parametrisation. The ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ has parametric equations

$$x = 3\cos t, \quad y = 2\sin t \quad (0 \le t \le 2\pi).$$

See Chapter A2, Section 5, for parametrisations of conics. Note that the letter t is used here for a parameter, as well as for a translation function.

Since E is obtained by translating this ellipse two units to the right and one unit up (see Activity 1.3), it follows that E has parametric equations

$$x = 2 + 3\cos t, \quad y = 1 + 2\sin t \quad (0 \le t \le 2\pi).$$

So, as t ranges through the interval $[0, 2\pi]$, the point

$$(x, y) = (2 + 3\cos t, 1 + 2\sin t)$$

traces out the ellipse E.

Now imagine that a processor supplies the points for this tracing process. The processor accepts a number t from the interval $[0, 2\pi]$, calculates the point $(2 + 3\cos t, 1 + 2\sin t)$ using the rule $t \longmapsto (2 + 3\cos t, 1 + 2\sin t)$,

and then outputs the result to be plotted in $\mathbb{R}^2$. Therefore this tracing process can be specified by a function, p say, with domain $[0, 2\pi]$ and codomain $\mathbb{R}^2$, as follows:

$$\begin{aligned} p : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (2 + 3\cos t, 1 + 2\sin t). \end{aligned}$$

This function p is called a *parametrisation function* for E. By calculating the image under p of any number in the domain $[0, 2\pi]$, we obtain a point on E; see Figure 1.5. For example, one point of E is given by

$$p(\tfrac{1}{2}\pi) = (2 + 3\cos(\tfrac{1}{2}\pi), 1 + 2\sin(\tfrac{1}{2}\pi)) = (2, 3).$$

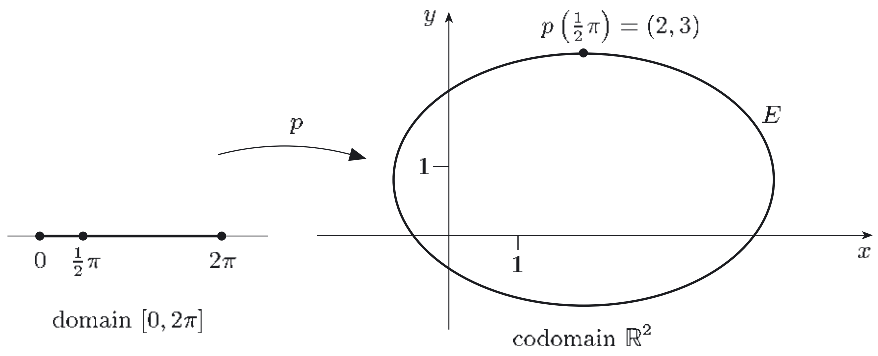


Figure 1.5 A parametrisation function for E

By calculating enough of these images we can plot E (or get a computer to plot E) to any desired accuracy.

Activity 1.4 Using a parametrisation function

For the above parametrisation function p, determine the images $p(0)$, $p(\pi)$ and $p(\frac{4}{3}\pi)$, and indicate them on a sketch of E. Describe the image set of p.

A solution is given on page 48.

The idea of a parametrisation function generalises as follows.

The Greek letter Γ, capital gamma, is often used to denote a curve.

Suppose that $p : I \longrightarrow \mathbb{R}^2$ is a function whose domain I is an interval of $\mathbb{R}$, and whose image set is a curve Γ in $\mathbb{R}^2$. Then we say that p is a **parametrisation function** for Γ.

The idea behind this definition is illustrated in Figure 1.6. As the parameter t ranges through the interval I, its image $p(t)$ traces out the curve Γ in $\mathbb{R}^2$.

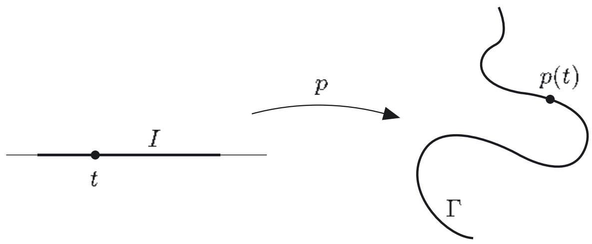


Figure 1.6 A parametrisation function for a curve Γ

Given a pair of parametric equations for a curve, we can define a parametrisation function for the curve. For example, the line segment joining the point $(4, 3)$ to the point $(7, 6)$ has parametric equations

See MST121, Chapter A2, Section 4.

$$x = 4 + 3t, \quad y = 3 + 3t \quad (0 \le t \le 1).$$

Therefore one possible parametrisation function for the line segment is

$$\begin{aligned} p : [0,1] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (4+3t, 3+3t). \end{aligned}$$

Activity 1.5 Finding a parametrisation function

Write down a parametrisation function for each of the following curves.

(a) the circle $(x-2)^2 + (y-5)^2 = 4$

(b) the parabola $y^2 = 8x$

(c) the right branch of the hyperbola $\frac{1}{3}x^2 - \frac{1}{10}y^2 = 1$

Solutions are given on page 48.

1.3 Functions of two variables

This subsection will not be assessed.

So far you have seen

- ◇ a function for which $\mathbb{R}^2$ is both the domain and codomain (the translation function t),
- ◇ a function for which $\mathbb{R}^2$ is the codomain but not the domain (the parametrisation function p).

We now examine a function for which $\mathbb{R}^2$ is the domain, but not the codomain. Such functions frequently arise when we measure some attribute of a point. For example, suppose we are interested in the distance from any given point (x, y) to the origin, as shown in Figure 1.7.

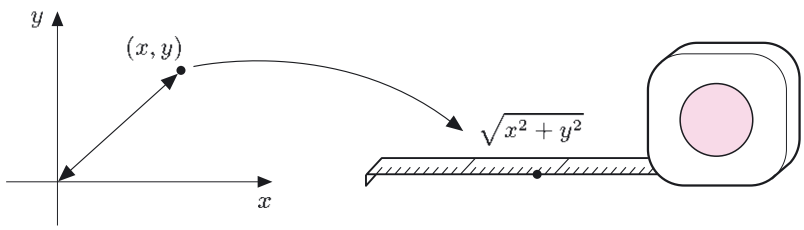


Figure 1.7 Distance to the origin

Pythagoras' Theorem tells us that this distance is given by the expression $\sqrt{x^2+y^2}$. What kind of *function* would give this distance?

Once again, imagine that a processor is doing the calculation. The processor has to accept any point (x, y) from $\mathbb{R}^2$, so this set is the domain, calculate the distance $\sqrt{x^2+y^2}$ using the rule $(x,y) \longmapsto \sqrt{x^2+y^2}$, and then output a non-negative real number. Since the output is non-negative, the codomain could be $[0, \infty)$ or $\mathbb{R}$.

So, if we choose $\mathbb{R}$ as the codomain, for simplicity, and d for the name of this function, then we obtain

$$\begin{aligned} d:\mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \sqrt{x^2+y^2}. \end{aligned}$$

The distance of any point to the origin can now be expressed as $d(x,y)=\sqrt{x^2+y^2}$, which is the image of the point (x,y) under d. For example, the distance of the point $(-4,3)$ to the origin is

$$d(-4,3)=\sqrt{(-4)^2+3^2}=5.$$

A function such as d whose domain is a subset of $\mathbb{R}^2$ and whose codomain is a subset of $\mathbb{R}$ is called a **function of two variables**. In certain circumstances, such functions can help to determine the 'type' of a given quadratic curve.

To see why this is so, first recall that we often represent a function of one real variable – that is, a real function – by plotting its graph. A function f of two real variables can also be represented by a graph, but it has to be drawn in three-dimensional space to accommodate the extra domain variable. We imagine that the domain of the function is part of the (x,y)-plane, which is shown horizontal in Figure 1.8. The z-axis passes through the origin of this horizontal plane and is vertical. Every point in this three-dimensional space can be represented by an ordered triple of coordinates (x,y,z), where z is the vertical coordinate; the mathematical notation for three-dimensional space is $\mathbb{R}^3$.

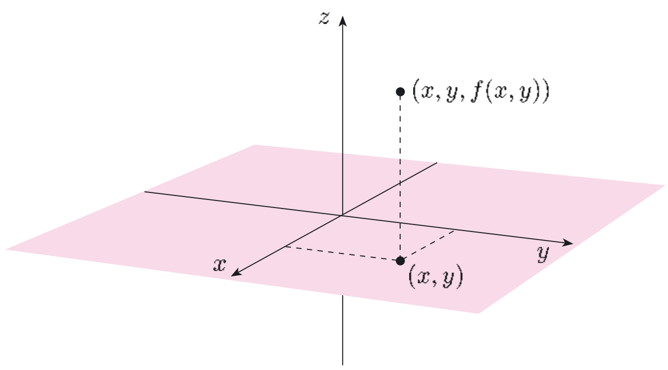


Figure 1.8 Constructing the graph of a function of two variables

For each point (x,y) in the domain of the function f, we travel in a direction parallel to the z-axis and at height $f(x,y)$ we place a point $(x,y,f(x,y))$. Doing this for each (x,y) in the domain, we obtain the graph of the function f as a *surface* lying over the domain.

Figure 1.9(a) shows the surface for the graph of the distance function d, for which $d(x,y)=\sqrt{x^2+y^2}$. It consists of a right circular cone with its vertex at the origin. The axis of the cone is aligned along the z-axis.

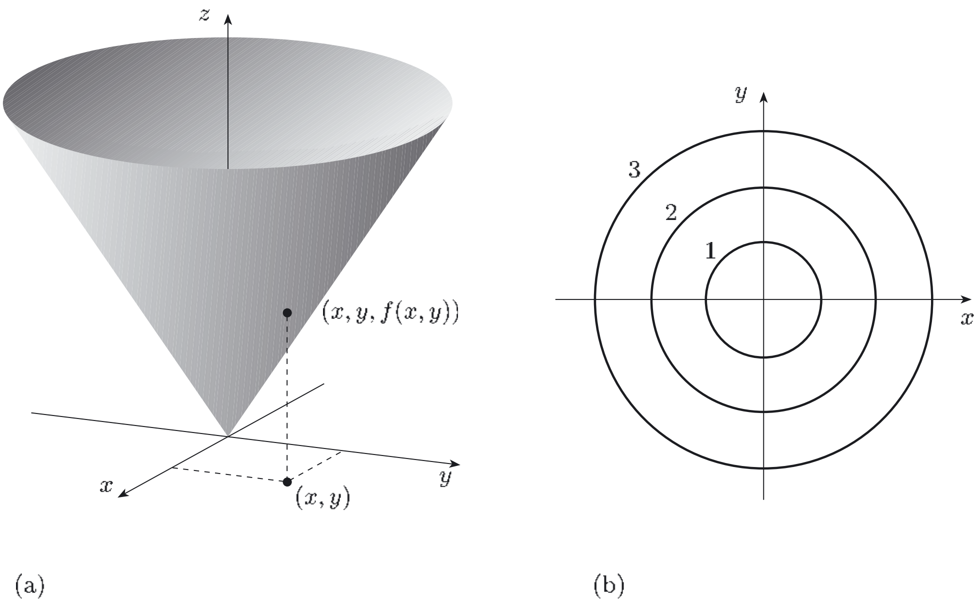


Figure 1.9 Surface and contour plots of the distance function

The graph of a function of two variables is sometimes called a **surface plot**. Closely related to this is an alternative type of plot, known as a *contour plot*. A **contour** consists of those points in the domain of a function f of two variables at which f takes a particular value, the height of the contour. A **contour plot** is a collection of such contours; a contour plot for the distance function d is shown in Figure 1.9(b).

A contour plot is like the collection of contours that appear on an Ordnance Survey map.

But what has all this got to do with the ellipse E? Well, E has equation

$$4x^2 + 9y^2 - 16x - 18y - 11 = 0.$$

So, if we define the function

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x, y) &\longmapsto 4x^2 + 9y^2 - 16x - 18y - 11, \end{aligned}$$

then we can rewrite the equation of E in the form

$$f(x, y) = 0.$$

Thus the ellipse E is the contour of the function f with height 0.

Therefore, one way to find the shape of E is to examine a contour plot of the function f, if this is available. A computer-generated contour plot of f is given in Figure 1.10, overleaf. The axes are not shown but the origin is at the centre of the plot.

See Section 5 for information on how to produce such a contour plot.

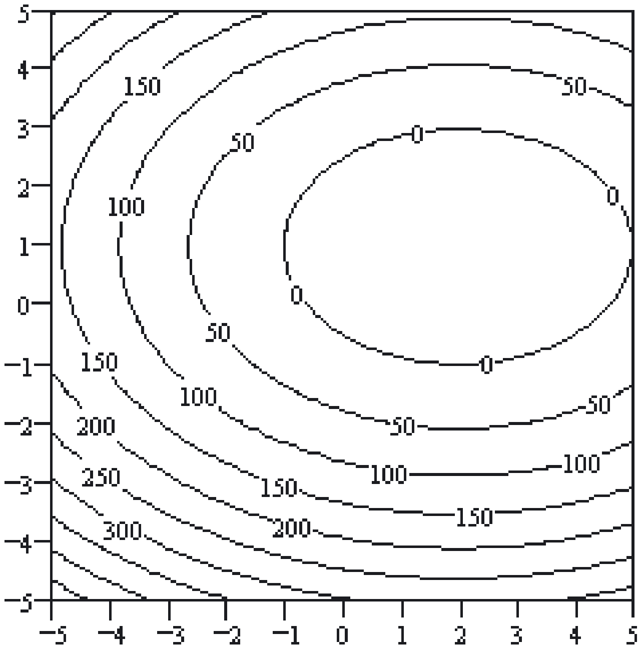


Figure 1.10 Contour plot of f for various heights including 0 (the ellipse E)

Although we cannot be sure of the *exact* position, size and orientation of the contours here, it does appear that they are all ellipses. Of course the plot does not provide us with the detailed information that is supplied by the parametrisation function p of E, but it does provide a useful check on the overall shape.

Summary of Section 1

This section has introduced:

◇ the general definition of a function;

◇ the concepts of domain, codomain, rule, image and image set in the context of examples of functions derived from geometry;

◇ the definition of a translation function, and a parametrisation function;

◇ the graph, or surface plot, of a function of two variables;

◇ contour plots and their relationship to the shapes of curves.

Exercises for Section 1

Exercise 1.1

This exercise concerns the quadratic curve L with equation

$$x^2 - y^2 - 4x + 6y = 6.$$

(a) Find the translation function that maps the quadratic curve L onto a conic K in standard position. Write down the equation of K.

(b) Write down the translation function that maps K onto L.

(c) Find a parametrisation function (or functions) for the quadratic curve L.

2 Isometries

To study Subsection 2.1, you will need a DVD player and DVD00095.

2.1 What is an isometry?

In geometry there are many occasions when we need to change the position and orientation of figures. For example, we may decide to investigate whether two triangles have the same shape and size by checking whether it is possible to superimpose one triangle on top of the other.

Again, we may wish to investigate which changes in position and orientation leave the figure looking the same. Or we may decide to investigate the ways in which motifs can be moved around the plane to form attractive patterns like the one shown in Figure 2.1.

Here, and later, minor differences in shape between the motifs are ignored. Colour is also ignored.

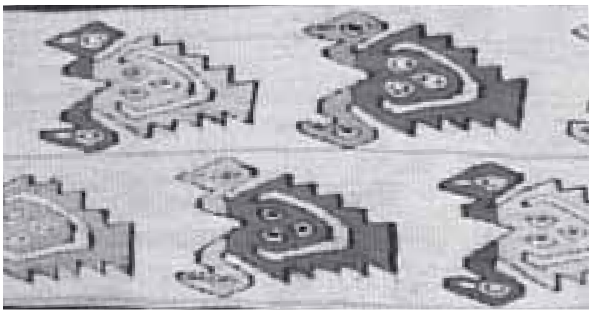


Figure 2.1 Patterns in fabric formed from a duck motif

In each case the idea is to move figures around the plane without changing their size or shape. But how can this be expressed mathematically? A clue is provided by the work in Section 1 where we used a translation function to move the conic E. Under translation, distances between points remain fixed, and this ensures that figures (like the conic E) move *rigidly* around the plane – that is, without changing their size and shape.

Functions with domain and codomain $\mathbb{R}^2$, such as translation functions, are often called transformations.

In general, we can ensure that a function preserves the size and shape of figures by restricting our attention to those functions that preserve distances between points. Such functions are called *isometries* from the Greek words 'iso', meaning 'same', and 'metron', meaning 'measure'.

> A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a (**plane**) **isometry** if the distance between any two points P and Q is equal to the distance between their images $f(P)$ and $f(Q)$.

Isometries are often described as **rigid-body motions**. For example, the image of a square under any isometry is a congruent square.

It turns out that every (plane) isometry is one of four basic types; a *translation*, a *rotation,* a *reflection*, or a *glide-reflection*. In this section these four types are examined, and you will see how they can be expressed as functions that map points of $\mathbb{R}^2$ to points of $\mathbb{R}^2$.

Now watch band A(v), 'Visualising isometries'.

The video analyses the patterns in fabrics, such as the one in Figure 2.1.

2.2 Translations

As you have seen, a translation (of the plane) is a function that moves each point of the plane a fixed distance in a fixed direction. For example, the translation function t in Section 1 that sends the ellipse E to standard position moves each point 2 units to the *left* and 1 unit *down* (see Figure 2.2), and is therefore defined by

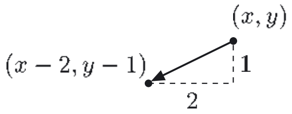


Figure 2.2

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x-2, y-1). \end{aligned}$$

For brevity we shall often write such functions in the form $t_{-2,-1}$, where the two subscripted numbers indicate the horizontal and vertical translations, respectively, the signs of the subscripts indicating their direction. Thus, the following function defines a translation of 6 units to the *right* and 4 units *up* (see Figure 2.3).

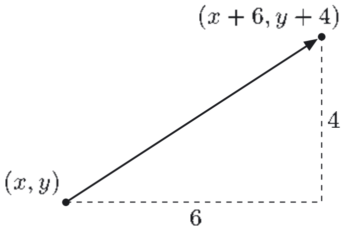


Figure 2.3

$$\begin{aligned} t_{6,4} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x+6, y+4). \end{aligned}$$

In general, we have the following two-line notation for a translation.

> A translation by p units horizontally and q units vertically has the form
>
> $$\begin{aligned} t_{p,q} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x+p, y+q). \end{aligned}$$

When showing the effects of isometries in sketches, it is important to use equal scales on the axes.

A convenient way to visualise the behaviour of a translation function is to sketch two copies of the plane $\mathbb{R}^2$, the domain and codomain, and indicate the positions of particular points before and after the translation has been applied. For example, Figure 2.4 shows the effect that $t_{6,4}$ has on the square S with vertices at the points $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

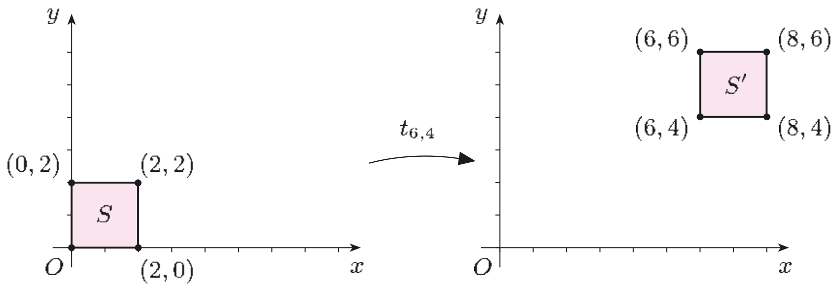


Figure 2.4 Translating a square

Since each point of the square S is displaced 6 units to the right and 4 units up, the overall effect is to move the entire square 6 units to the right and 4 units up. This produces a new square S' with vertices at the points $(6,4)$, $(8,4)$, $(8,6)$, $(6,6)$. In other words, S' is the image of S under $t_{6,4}$; thus we can write

$$S' = t_{6,4}(S).$$

Activity 2.1 Translating a triangle

(a) Define a translation function that moves points 3 units to the right and 2 units down.

(b) Draw a sketch to illustrate the effect that the function has on the triangle T with vertices at $(2,1)$, $(5,3)$ and $(3,4)$.

Solutions are given on page 48.

Now suppose that we want to reverse, or undo, the effect of the translation $t_{6,4}$ shown in Figure 2.4, to restore the square to its original position. By definition $t_{6,4}$ moves points 6 units to the right and 4 units up, so to undo its effect we need the translation $t_{-6,-4}$ that moves points 6 units to the *left* and 4 units *down*. Indeed, it is easy to check that

$$\begin{aligned} t_{-6,-4} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x-6, y-4) \end{aligned}$$

sends $(6,4)$, $(8,4)$, $(8,6)$, $(6,6)$ back to $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$, respectively, so the square is restored to its original position (see Figure 2.5). We say that the translation $t_{-6,-4}$ is the *inverse* of the translation $t_{6,4}$.

The notion of the inverse function of a *real* function is introduced in MST121 Chapter A3, Section 4.

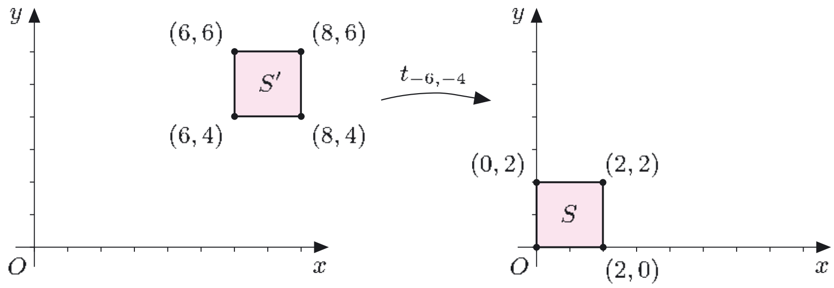


Figure 2.5 The inverse translation $t_{-6,-4}$

In general, the inverse of a translation is obtained by changing the signs of the numbers in the subscript.

> The **inverse** of the translation $t_{p,q}$ is the translation $t_{-p,-q}$.

Activity 2.2 Inverse of a translation

(a) Write down the inverse of the translation described in Activity 2.1.

(b) Check that this inverse maps the image of the triangle in Activity 2.1 to its original position.

Solutions are given on page 49.

Next we consider the effect of following one translation by another. Figure 2.6 illustrates the effect of first using $t_{-4,0}$ to translate a duck motif 4 units to the left and then using $t_{1,1}$ to translate the result 1 unit to the right and 1 unit up. Overall, the duck motif is sent 3 units to the left and 1 unit up, a movement that could be achieved directly by using the single translation $t_{-3,1}$.

Once again, minor differences between the motifs are ignored.

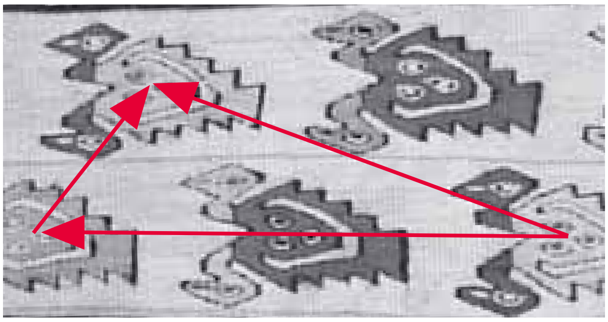


Figure 2.6 A composite translation

The translation $t_{-3,1}$ is called the *composite* of $t_{-4,0}$ followed by $t_{1,1}$. In symbols, this is written

$$t_{-3,1} = t_{1,1} \circ t_{-4,0}.$$

Note that the first translation to be performed is written to the right of the symbol $\circ$, which is called the **composition** operation. In fact, the composite $t_{-4,0} \circ t_{1,1}$ has the same effect as $t_{1,1} \circ t_{-4,0}$; that is, $t_{-4,0} \circ t_{1,1} = t_{-3,1}$. In general, the order in which two translations are composed does not matter, and so the phrase 'the composite of two translations' is not ambiguous.

The symbol $\circ$ is read as 'oh' or as 'circle'.

Activity 2.3 Composing translations

Draw a diagram to illustrate the effect on a triangle of following the translation $t_{3,6}$ by the translation $t_{1,-2}$. Hence describe the composite $t_{1,-2} \circ t_{3,6}$ in words and symbols.

A solution is given on page 49.

These examples illustrate that the composite of any two translations is simply the translation obtained by adding the corresponding numbers in the subscripts.

> The **composite** of two translations $t_{p,q}$ and $t_{r,s}$ is given by
>
> $$t_{p+r,q+s}.$$

2.3 Rotations

A **rotation** (of the plane) about a fixed point is a function that moves each point through a fixed angle about the fixed point; see Figure 2.7.

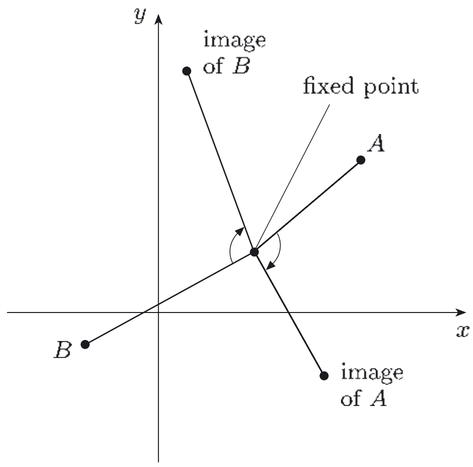


Figure 2.7 Images under a (clockwise) rotation

In this subsection we concentrate on rotations about the origin. The notation for the function that rotates points of the plane (about the origin) through the angle θ is r_θ (see Figure 2.8). The rotation is taken to be anticlockwise if θ is positive and clockwise if θ is negative. Such a function is called a **rotation function**, or simply a 'rotation'. Before rotations are defined algebraically, it is helpful to consider an example.

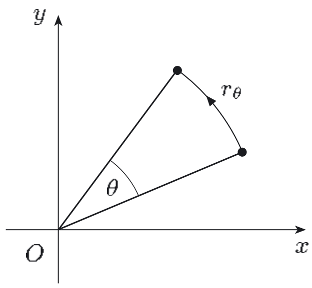


Figure 2.8 The rotation r_θ

Activity 2.4 Finding an image under a rotation

(a) Sketch the triangle T with vertices $(0,0)$, $(3,0)$, $(3,2)$, and the image T' of T under the rotation $r_{\pi/2}$.

(b) What is the image of $(3,2)$ under the rotation $r_{\pi/2}$?

A solution is given on page 49.

Figure 2.9 indicates what happens to the coordinates of an *arbitrary* point P under the rotation $r_{\pi/2}$. Prior to rotation (left) P has coordinates (x, y). The triangle shown, which has vertices $(0,0)$, $(x,0)$ and (x,y), has the same shape after the rotation (right), but its base ends up along the y-axis. Thus P ends up at a new location P' with coordinates $(-y, x)$.

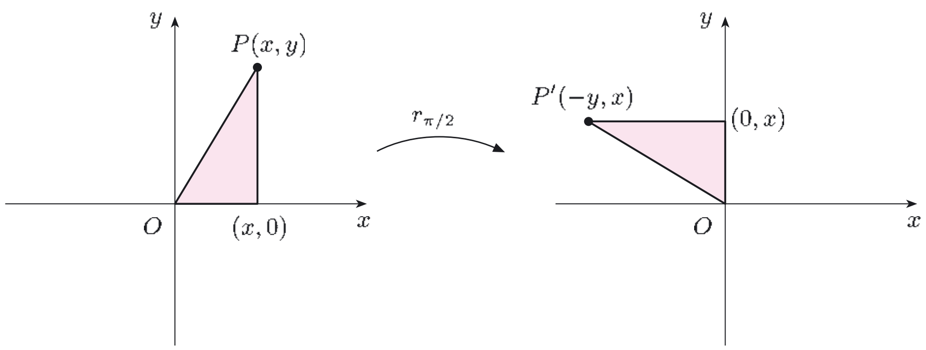


Figure 2.9 Rotating a point using $r_{\pi/2}$

These observations enable us to define the rotation function $r_{\pi/2}$ as follows:

$$\begin{aligned} r_{\pi/2} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-y, x). \end{aligned}$$

For example, $r_{\pi/2}(7, 3) = (-3, 7)$.

Activity 2.5 Finding images under a rotation

Find the images of the points $(-3, 2)$, $(1, 0)$, $(0, 1)$ and $(-2, -3)$ under the rotation $r_{\pi/2}$. Sketch your results on a single diagram.

A solution is given on page 49.

Now consider a general rotation r_θ of the plane about the origin. To be specific, we concentrate on a point P in the first quadrant with coordinates (u, v), as shown in Figure 2.10, and consider what happens to P after it has been rotated anticlockwise through an angle θ about the origin O. To help us keep track of P we have shaded a right-angled triangle PQO whose base lies along the x-axis and whose hypotenuse joins the origin to P. Under the rotation both the point P and the right-angled triangle rotate through the angle θ, to produce the triangle $P'Q'O$. In Figure 2.10, the angle θ is chosen so that triangle $P'Q'O$ is in the first quadrant. Thus OQ' is inclined at angle θ to the horizontal, and side $P'Q'$ is inclined at angle θ to the vertical. Since the shape and size of the triangle is unchanged by the rotation, it follows that OQ' has length u and $Q'P'$ has length v.

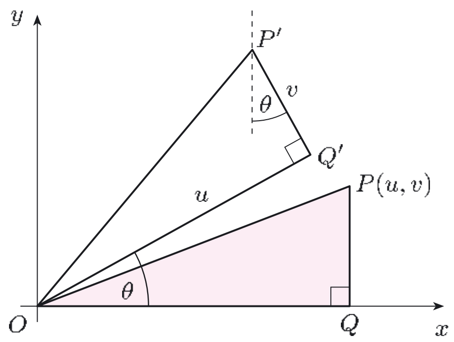


Figure 2.10 Rotating a point P

The next step is to project the sides OQ' and $P'Q'$ onto the x-axis, as indicated by the dotted lines in Figure 2.11.

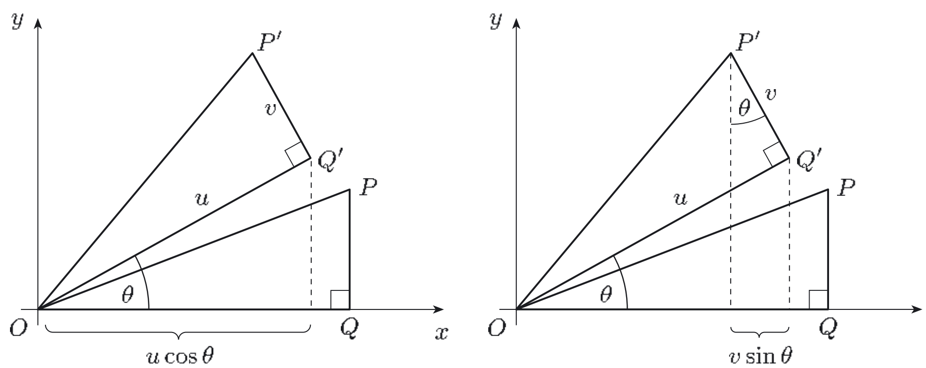


Figure 2.11 Finding the coordinates of P'

The projection of OQ' has length $u\cos\theta$, and the projection of $P'Q'$ has length $v\sin\theta$. The x-coordinate of the point P' is therefore $u\cos\theta - v\sin\theta$.

Activity 2.6 Determining the y-coordinate of P'

By following the steps of the previous argument, with suitable adjustments, determine the y-coordinate of the point P'.

A solution is given on page 49.

The conclusion is that after a rotation about the origin through an angle θ, the point (u, v) ends up at the point with coordinates

$$(u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta).$$

In fact, this result is true for an *arbitrary* point (u, v) in the plane and for *any* angle θ.

Thus we have obtained the following two-line notation for a rotation.

> A rotation about the origin, through an angle θ, has the form
>
> $$\begin{aligned} r_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

Remember that the angle for an anticlockwise rotation is positive, whereas the angle for a clockwise rotation is negative.

For example, a clockwise rotation through an eighth of a full turn about the origin corresponds to a turn through the angle $\theta = -\pi/4$. Since

A full turn is a complete revolution.

$$\cos\left(-\tfrac{1}{4}\pi\right) = \tfrac{1}{2}\sqrt{2} = 1/\sqrt{2} \quad \text{and} \quad \sin\left(-\tfrac{1}{4}\pi\right) = -\tfrac{1}{2}\sqrt{2} = -1/\sqrt{2},$$

it follows that the rotation function is

$$\begin{aligned} r_{-\pi/4} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \left(\frac{x+y}{\sqrt{2}}, \frac{-x+y}{\sqrt{2}}\right). \end{aligned}$$

Activity 2.7 Rotations through a given angle

For each of the following angles, determine in two-line notation the function that specifies the rotation through that angle about the origin. In each case, check your result by calculating the image of $(1, 0)$.

(a) π (b) $\frac{1}{4}\pi$ (c) $-\frac{2}{3}\pi$

Solutions are given on page 50.

Earlier you saw that the effect of a translation $t_{p,q}$ can be undone by the application of the inverse translation $t_{-p,-q}$. A similar property holds for rotations about the origin.

> The **inverse** of the rotation r_θ is the rotation $r_{-\theta}$.

As for translations, we can consider the effect of following one rotation by another. For example, Figure 2.12 illustrates the effect of first using $r_{2\pi/3}$ to rotate a motif $\frac{2}{3}\pi$ radians anticlockwise about its centre and then using $r_{\pi/3}$ to rotate the image another $\frac{1}{3}\pi$ radians anticlockwise. Overall, the motif rotates π radians anticlockwise, which could be achieved directly by using the single rotation r_π.

Here the centre of the motif is taken to be the origin.

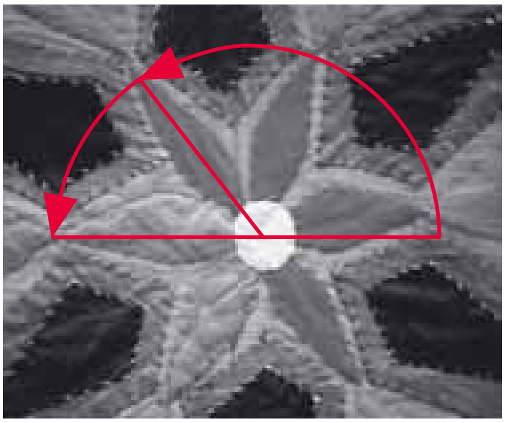


Figure 2.12 A composite rotation

The rotation r_π is the *composite* of $r_{2\pi/3}$ followed by $r_{\pi/3}$, and we write

$$r_\pi = r_{\pi/3} \circ r_{2\pi/3}.$$

In general, if we rotate the plane about the origin through an angle ϕ and follow this by a rotation of the plane about the origin through a further angle θ, then the overall effect is a rotation of the plane about the origin through the angle sum $\theta + \phi$. The same overall effect is obtained if we rotate first through θ and then through ϕ; that is,

$$r_\theta \circ r_\phi = r_\phi \circ r_\theta.$$

These observations hold whatever the signs of the angles.

> The **composite** of two rotations r_ϕ and r_θ about the origin is given by
>
> $$r_{\theta+\phi}.$$

Since a full turn corresponds to 2π radians, the rotation r_θ is in effect the same rotation as $r_{\theta+2\pi}$ and $r_{\theta-2\pi}$. It is usual to express the results of calculations involving rotations in terms of angles in a chosen interval of length 2π. A common choice is $(-\pi, \pi]$. Working with this interval, we obtain

Another common choice is $[0, 2\pi)$.

$$r_{-2\pi/3} \circ r_{-\pi/2} = r_{-7\pi/6} = r_{5\pi/6},$$

since $-7\pi/6$ lies outside $(-\pi, \pi]$, but $-7\pi/6 + 2\pi = 5\pi/6$ lies in it.

Activity 2.8 Composing rotations about the origin

Determine each of the following composites, ensuring that the angle in your answer lies in the interval $(-\pi, \pi]$.

(a) $r_{\pi/2} \circ r_{\pi/3}$ (b) $r_{2\pi/3} \circ r_{\pi/2}$ (c) $r_{\pi/3} \circ r_{-\pi/2}$

Solutions are given on page 50.

2.4 Reflections

Imagine that a mirror is held vertically along the central line of the cloth design in Figure 2.13. The animal motif on the right can then be regarded as the mirror image of the animal motif on the left. Moreover, if the mirror is two-sided, the animal motif on the left can be regarded as the mirror image of the animal motif on the right.

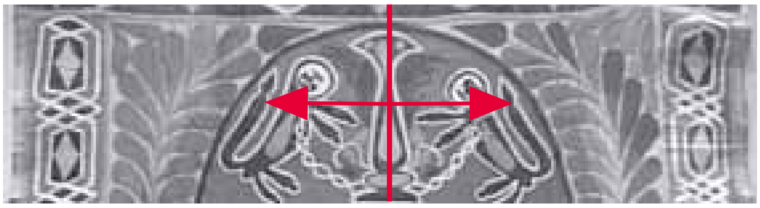


Figure 2.13 Cloth design with animal motifs

This suggests the following (mathematical) definition of **reflection in a line** ℓ. Each point P of the plane is sent to a point P' on the other side of ℓ in such a way that ℓ is the perpendicular bisector of the line segment PP', as shown in Figure 2.14(a). Points on ℓ remain fixed.

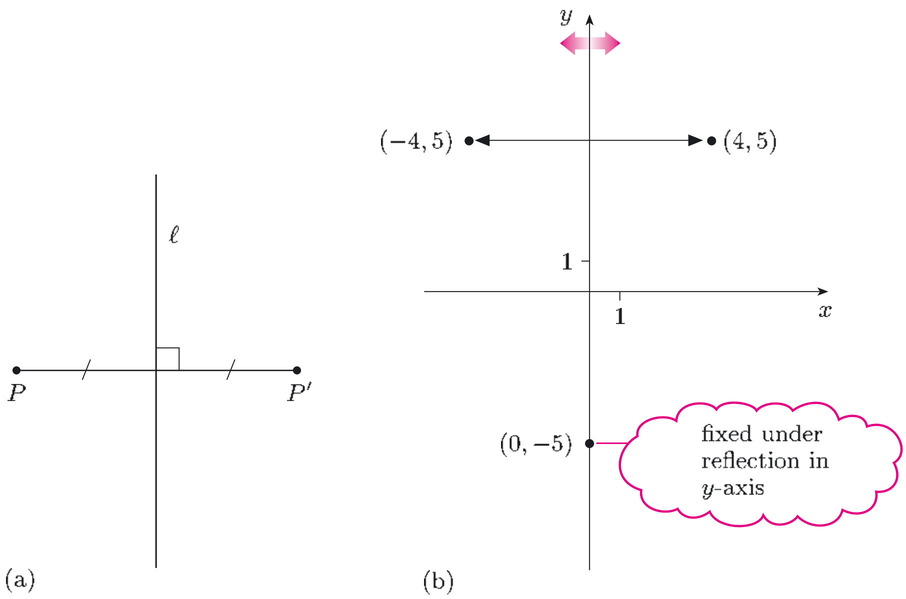


Figure 2.14 Reflections

For example, reflection in the y-axis sends the point $(4, 5)$ to $(-4, 5)$, and it sends $(-4, 5)$ to $(4, 5)$; see Figure 2.14(b). Under this reflection, points on the y-axis, like $(0, -5)$, remain fixed. In general, an arbitrary point (x, y) is sent across the y-axis to $(-x, y)$. We can therefore write the reflection in the y-axis in the form of a function as follows:

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-x, y). \end{aligned}$$

The function f is an example of a **reflection function**, or simply a 'reflection'.

Activity 2.9 Some reflections

For each of the following reflections, sketch the images of the points $(1, 0)$, $(0, 1)$, $(0, 0)$, $(1, 1)$, and then use two-line notation to write down a function f that describes the reflection.

(a) a reflection in the x-axis

(b) a reflection in the line $y = x$

(c) a reflection in the line $y = -x$

Solutions are given on page 50.

As for translations and rotations, we can consider the function that undoes the effect of a reflection. A moment's thought shows that any reflection is its own inverse; that is, it is **self-inverse**.

All reflections are self-inverse.

In Subsection 2.6, an algebraic rule is given for reflection in a line through the origin at a given angle, but first there is a little more work to do on composite functions.

2.5 Composite isometries

So far we have seen that the composite of two translations is a translation, and the composite of two rotations about the origin is a rotation about the origin. In this subsection we examine what happens when we compose different types of isometries.

Activity 2.10 Composing two isometries

In the following design, consider the process of superimposing one of the leaf motifs onto its right-hand neighbour. Try to describe this process as a composite of two isometries.

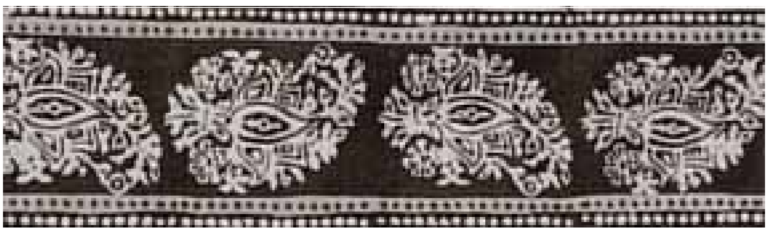


Figure 2.15 A leaf motif design

A solution is given on page 51.

The isometry described in Activity 2.10 is an example of a new kind of isometry known as a glide-reflection. A **glide-reflection** in a line ℓ is defined to be reflection in ℓ followed by a translation parallel to ℓ.

The same glide-reflection is obtained by performing the translation first and then the reflection.

As an example, consider the glide-reflection obtained by following the reflection q in the x-axis by the translation t through 1 unit in the positive direction of the x-axis. The definitions of q and t are as follows.

$$\begin{aligned} q\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x,-y) \end{aligned} \quad \text{and} \quad \begin{aligned} t\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+1,y). \end{aligned}$$

Figure 2.16 illustrates the overall effect of the glide-reflection on a particular right-angled triangle. First q reflects the triangle across the x-axis, and the resulting triangle is then translated 1 unit to the right by t.

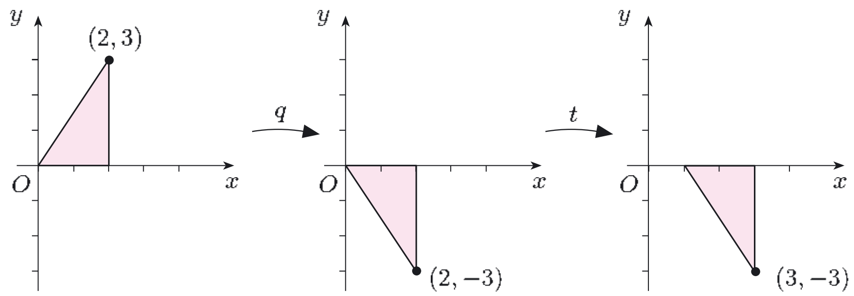


Figure 2.16 A glide-reflection

Although the glide-reflection is defined in terms of two isometries, it is still an isometry in its own right, and so we should be able to determine its rule algebraically. In order to do this it is helpful to return to the 'processor' interpretation of a function. Here, there are two processors, one for t and one for q. To follow q by t, the output of q is used as input for t, thereby producing the 'composite processor' illustrated by the outer box in Figure 2.17. For example, if the point $(2,3)$ is fed into this composite processor, then it is first processed by q to produce the intermediate point $q(2,3) = (2,-3)$, and this is then processed by t to produce the final output $t(2,-3) = (3,-3)$.

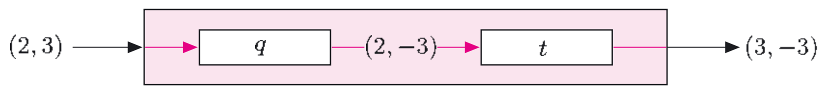


Figure 2.17 The composite processor

Similarly, if an arbitrary point (x,y) is fed into the composite processor, then it is first processed by q to produce the intermediate point $q(x,y) = (x,-y)$, and this is then processed by t to produce the final output. The effect of t is to add 1 to the first coordinate of its input and leave the second coordinate unchanged. Thus we obtain

$$t(q(x,y)) = t(x,-y) = (x+1,-y).$$

Overall we obtain a function, called the *composite* of q followed by t, with domain and codomain $\mathbb{R}^2$. The function is denoted by $t \circ q$, and under it each point (x,y) in the domain is mapped to the point $(x+1,-y)$. So the rule for the function is $(x,y) \longmapsto (x+1,-y)$.

The required glide-reflection is therefore given in two-line form by

$$\begin{aligned} t \circ q : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+1,-y). \end{aligned}$$

As you can check, the composite $q \circ t$ also gives this glide-reflection.

We have just composed a reflection with a translation to produce a glide-reflection, but it is possible to compose any two isometries in a similar way, as illustrated in Figure 2.18.

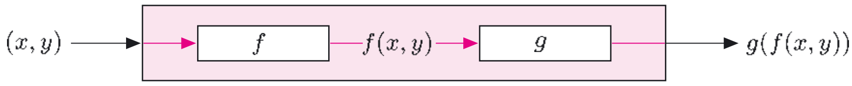


Figure 2.18 Composing isometries

When a point (x, y) is fed into the composite processor it is first processed by f to produce the intermediate point $f(x, y)$. This is then processed by g to produce the final output $g(f(x, y))$.

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be two isometries of the plane. Then the **composite** $g \circ f$ (**of** f **followed by** g) is the isometry defined by

$$\begin{aligned} g \circ f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto g(f(x, y)). \end{aligned}$$

The next activity asks you to compose a reflection and a rotation, in both possible orders.

Activity 2.11 Composing isometries

Let q be the transformation that reflects the plane in the x-axis and let r be the transformation that rotates the plane clockwise about the origin through a quarter of a turn. Algebraically these have the forms

In terms of the notation r_θ, the rotation r is $r_{-\pi/2}$.

$$\begin{aligned} q\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x, -y) \end{aligned} \quad \text{and} \quad \begin{aligned} r\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (y, -x). \end{aligned}$$

Determine the composite isometries $r \circ q$ and $q \circ r$, and interpret them geometrically. Is it true that $r \circ q = q \circ r$?

Solutions are given on page 51.

Comment

This activity shows that the *order* in which functions are composed can make a difference to the result.

2.6 Reflections revisited

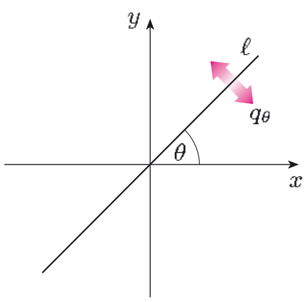


Figure 2.19 The reflection q_θ

In terms of the notation q_θ, the reflection q is q_0.

We now return to the problem of finding the rule for reflection in a general line ℓ through the origin. We shall assume that ℓ makes an angle θ with the positive x-axis, and we shall denote the reflection function by q_θ as shown in Figure 2.19.

The idea is to express q_θ as a composite of isometries with rules that we already know. The triangles in Figure 2.20 show that the reflection in ℓ can be expressed as the composite of a clockwise rotation through the angle 2θ, followed by a reflection q in the x-axis; that is, $q_\theta = q \circ r_{-2\theta}$. The rotation $r_{-2\theta}$ sends ℓ to a line ℓ', and this is sent back to ℓ by the reflection q. Overall ℓ stays fixed but the plane has been reflected in ℓ, as required.

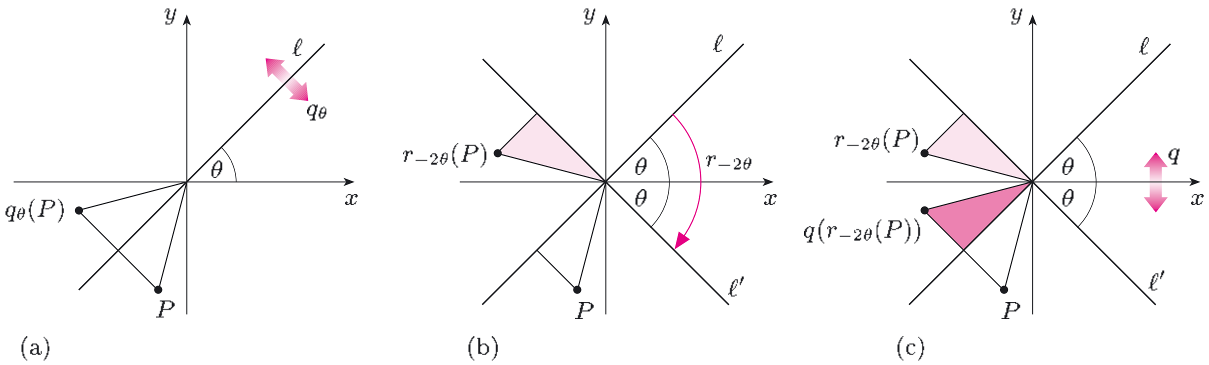


Figure 2.20 Showing that $q_\theta = q \circ r_{-2\theta}$

The rule for a *clockwise* rotation through 2θ is

See Subsection 2.3.

$$r_{-2\theta}(x,y) = (x\cos(-2\theta) - y\sin(-2\theta), x\sin(-2\theta) + y\cos(-2\theta))$$

and the rule for reflection in the x-axis is

$$q(x,y) = (x,-y).$$

Recalling that $\cos(-2\theta) = \cos(2\theta)$ and $\sin(-2\theta) = -\sin(2\theta)$, we obtain

$$\begin{aligned} q(r_{-2\theta}(x,y)) &= q(x\cos(-2\theta) - y\sin(-2\theta), x\sin(-2\theta) + y\cos(-2\theta)) \\ &= q(x\cos(2\theta) + y\sin(2\theta), -x\sin(2\theta) + y\cos(2\theta)) \\ &= (x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta)). \end{aligned}$$

Since $q_\theta = q \circ r_{-2\theta}$, the reflection in ℓ is given in two-line notation by

$$\begin{aligned} q_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta)). \end{aligned}$$

Thus we have obtained the following two-line notation for a reflection.

> Let ℓ be a line that passes through the origin and makes an angle θ with the positive x-axis. Then the **reflection** in ℓ is given by
>
> $$\begin{aligned} q_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta)). \end{aligned}$$

Activity 2.12 Reflection in a given line

Let ℓ be the line through the origin that makes an angle of $\frac{1}{6}\pi$ with the positive x-axis. Determine the reflection $q_{\pi/6}$ in ℓ in two-line notation.

A solution is given on page 51.

Later in the course we consider the composite $q_\theta \circ q_\phi$, one reflection followed by another. It turns out that $q_\theta \circ q_\phi$ is always a rotation and that, in general,

$$q_\theta \circ q_\phi \neq q_\phi \circ q_\theta.$$

Summary of Section 2

This section has introduced:

- ◇ the two-line notation for a translation $t_{p,q}$;
- ◇ the two-line notation for a rotation r_θ through an angle θ about the origin;
- ◇ the two-line notation for a reflection q_θ in a line through the origin at an angle θ with the positive x-axis;
- ◇ the inverses of translations and rotations, and the fact that reflections are self-inverse;
- ◇ formulas for writing down the composite of two translations, or two rotations about the origin;
- ◇ the method for determining the composite of two isometries (for example, a glide-reflection).

Exercises for Section 2

Exercise 2.1

Express each of the following translations in the form $t_{p,q}$.

(a) the inverse of $t_{5,-3}$

(b) the composite $t_{2,-3} \circ t_{4,5}$

Exercise 2.2

Express each of the following rotations in the form r_θ, where θ lies in the interval $(-\pi, \pi]$.

(a) the inverse of $r_{2\pi/3}$

(b) the composite $r_{3\pi/5} \circ r_{4\pi/5}$

Exercise 2.3

Give a definition in two-line notation of each of the following isometries.

(a) a translation three units to the left and two units down

(b) a clockwise rotation through an angle of $\pi/3$ about the origin

(c) a reflection in the line $y = \sqrt{3}x$

Exercise 2.4

Determine the composite $g \circ f$, where

$$\begin{aligned} f\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (y,x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} g\colon \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+2, y+2). \end{aligned}$$

Give a geometric interpretation of f and g, and hence of $g \circ f$.

3 Trigonometric formulas

In this section we shall use the rotation functions introduced in Section 2 to derive some important relationships amongst trigonometric functions, needed for our further study of conics in Section 4.

The formulas in this section were known, in somewhat different form, to Ptolemy of Alexandria (c. 90–168 AD). Ptolemy was an astronomer, geographer and author of a textbook on trigonometry.

3.1 Basic trigonometric relationships

The basic formulas relating the sine, cosine and tangent of an angle θ are the definition of $\tan\theta$ and the 'Pythagorean identity':

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \text{and} \qquad \cos^2\theta + \sin^2\theta = 1; \tag{3.1}$$

see Figure 3.1, where the line OP makes the angle θ with the positive x-axis.

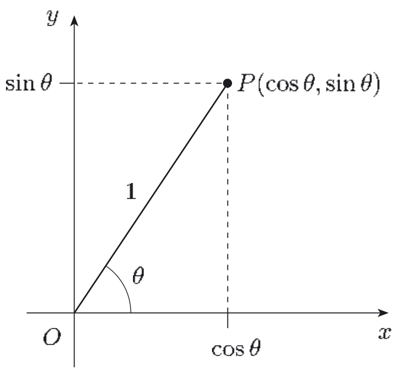


Figure 3.1 Sine and cosine of θ

If we know the exact value of any one of $\sin\theta$, $\cos\theta$ or $\tan\theta$, then equations (3.1) allow us to find the exact values of the other two, apart from the choice of $+$ or $-$ sign. This sign is found by checking in which quadrant the point P lies, and using Figure 3.2.

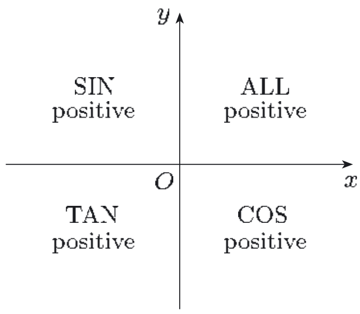


Figure 3.2 Signs of trigonometric functions

For example, if we know the value of $\sin\theta$, then we can find values for $\cos\theta$ and $\tan\theta$, as follows:

$$\cos^2\theta + \sin^2\theta = 1, \quad \text{so} \quad \cos^2\theta = 1 - \sin^2\theta;$$

hence

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}, \quad \text{so} \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}. \tag{3.2}$$

On the other hand, if we know the value of $\tan\theta$, then we can find values for $\sin\theta$ and $\cos\theta$, as follows:

$$\cos^2\theta + \sin^2\theta = 1, \quad \text{so} \quad 1 + \tan^2\theta = \frac{1}{\cos^2\theta};$$

hence

$$\cos\theta = \pm\frac{1}{\sqrt{1+\tan^2\theta}}, \quad \text{so} \quad \sin\theta = \tan\theta\cos\theta = \pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}. \tag{3.3}$$

Other approaches to these calculations are possible. For example, if we know the value of one of $\sin\theta$, $\cos\theta$ and $\tan\theta$ *and* we know that θ lies in the interval $(0, \frac{1}{2}\pi)$, then the values of the other two trigonometric ratios can be found by using an appropriate right-angled triangle. In some cases, the calculations involved are simple. For example, if $\sin\theta = \frac{4}{5}$, then from Figure 3.3, we obtain the exact values

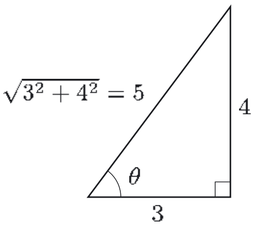


Figure 3.3 A 3-4-5 triangle

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \tfrac{3}{5}$$

and

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \tfrac{4}{3}.$$

In other cases the calculations are not as simple, as you will see in Activity 3.1(c).

Activity 3.1 Relationships between sine, cosine and tangent

(a) Find $\cos\theta$ and $\tan\theta$, given that $\sin\theta = \frac{1}{2}$ and θ lies in the interval $(\frac{1}{2}\pi, \pi)$.

(b) Find $\cos\theta$ and $\sin\theta$, given that $\tan\theta = -\frac{3}{4}$ and θ lies in the interval $(-\frac{1}{2}\pi, 0)$.

(c) Find $\sin\theta$ and $\tan\theta$, given that $\cos\theta = \frac{6}{7}$ and θ lies in the interval $(0, \frac{1}{2}\pi)$.

Solutions are given on page 51.

Finally in this subsection, we recall that the 'reciprocal' trigonometric functions secant, cosecant and cotangent are defined as follows:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

Given the value of any one of $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\operatorname{cosec}\theta$ or $\cot\theta$, we can find the values of all the others, apart from the choice of sign. For example, if we know that $\cot\theta = -\frac{4}{3}$, then $\tan\theta = 1/(-\frac{4}{3}) = -\frac{3}{4}$. We can then find $\sin\theta$ and $\cos\theta$ as in Activity 3.1(b), and hence find $\operatorname{cosec}\theta$ and $\sec\theta$.

The identity $\cos^2\theta + \sin^2\theta = 1$ can be rewritten in terms of the reciprocal trigonometric functions as follows:

Equation (3.4) was used in Chapter A2, Section 5 to obtain parametric equations for a hyperbola.

$$1 + \tan^2\theta = \sec^2\theta, \tag{3.4}$$

and

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta,$$

on dividing through by $\cos^2\theta$ and $\sin^2\theta$, respectively.

3.2 Sum and difference formulas

Next, two identities are derived which express the sine and cosine of the sum $\phi + \theta$ in terms of the sines and cosines of ϕ and θ. They are known as the *sum formulas*.

Suppose that the point P lies 1 unit from the origin, with OP making an angle ϕ with the positive x-axis, as shown in Figure 3.4. The coordinates of P are $(\cos\phi, \sin\phi)$.

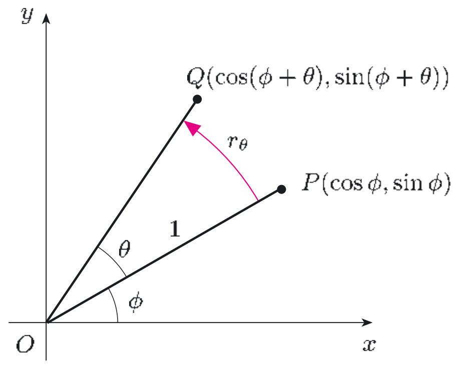


Figure 3.4 Rotating the point P

If we now apply the rotation r_θ to the point P, then we obtain the point Q that lies 1 unit from the origin, with OQ making an angle $\phi + \theta$ with the positive x-axis. The coordinates of Q are therefore $(\cos(\phi+\theta), \sin(\phi+\theta))$, as shown in Figure 3.4.

We can describe the effect of the rotation r_θ on P algebraically as

$$(\cos(\phi+\theta), \sin(\phi+\theta)) = r_\theta(\cos\phi, \sin\phi).$$

Using the formula $r_\theta(x, y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, with $x = \cos\phi$ and $y = \sin\phi$, we conclude that

See Subsection 2.3.

$$(\cos(\phi+\theta), \sin(\phi+\theta)) = (\cos\phi\cos\theta - \sin\phi\sin\theta, \cos\phi\sin\theta + \sin\phi\cos\theta).$$

The x-coordinates of this equation give the sum formula for cosine and the y-coordinates give the sum formula for sine. The formulas are recorded below, with that for tangent.

Sum formulas

$$\sin(\phi+\theta) = \sin\phi\cos\theta + \cos\phi\sin\theta \qquad (3.5)$$

$$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta \qquad (3.6)$$

$$\tan(\phi+\theta) = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \qquad (3.7)$$

The tangent sum formula is obtained as follows:

$$\begin{aligned}\tan(\phi+\theta) &= \frac{\sin(\phi+\theta)}{\cos(\phi+\theta)} && \text{(equations (3.1) with } \theta \text{ replaced by } \phi+\theta\text{)}\\ &= \frac{\sin\phi\cos\theta + \cos\phi\sin\theta}{\cos\phi\cos\theta - \sin\phi\sin\theta} && \text{(equations (3.5) and (3.6))}\end{aligned}$$

$$= \frac{\dfrac{\sin\phi}{\cos\phi} + \dfrac{\sin\theta}{\cos\theta}}{1 - \dfrac{\sin\phi\sin\theta}{\cos\phi\cos\theta}}$$

$$= \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}.$$

One application of the sum formulas is to use known exact values of sine, cosine and tangent to calculate other exact values. For example, since

$$\cos(\tfrac{1}{4}\pi) = \sin(\tfrac{1}{4}\pi) = \tfrac{1}{2}\sqrt{2}, \quad \cos(\tfrac{1}{3}\pi) = \tfrac{1}{2} \quad \text{and} \quad \sin(\tfrac{1}{3}\pi) = \tfrac{1}{2}\sqrt{3},$$

it follows from the cosine sum formula that

$$\begin{aligned}\cos\left(\tfrac{7}{12}\pi\right) &= \cos\left(\tfrac{3}{12}\pi + \tfrac{4}{12}\pi\right)\\ &= \cos\left(\tfrac{1}{4}\pi + \tfrac{1}{3}\pi\right)\\ &= \cos\left(\tfrac{1}{4}\pi\right)\cos\left(\tfrac{1}{3}\pi\right) - \sin\left(\tfrac{1}{4}\pi\right)\sin\left(\tfrac{1}{3}\pi\right)\\ &= \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2} - \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}\sqrt{3}\\ &= \tfrac{1}{4}\left(\sqrt{2} - \sqrt{6}\right).\end{aligned}$$

Activity 3.2 Using the sum formulas

Find the exact values of $\sin\left(\tfrac{7}{12}\pi\right)$ and $\tan\left(\tfrac{7}{12}\pi\right)$.

Solutions are given on page 52.

These sign formulas were discussed in MST121, Chapter A2, Section 3.

If we replace θ by $-\theta$ in the sum formulas and use the *sign formulas*,

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta, \quad \tan(-\theta) = -\tan\theta,$$

then we obtain the following *difference formulas*.

Difference formulas

$$\begin{aligned}\sin(\phi - \theta) &= \sin\phi\cos\theta - \cos\phi\sin\theta\\ \cos(\phi - \theta) &= \cos\phi\cos\theta + \sin\phi\sin\theta\\ \tan(\phi - \theta) &= \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}\end{aligned}$$

Once again we can use these formulas to determine new exact values of sine, cosine and tangent from known ones.

Activity 3.3 Using the difference formulas

Find the exact values of $\sin(\tfrac{1}{12}\pi)$, $\cos(\tfrac{1}{12}\pi)$ and $\tan(\tfrac{1}{12}\pi)$.

Solutions are given on page 52.

3.3 Double-angle and half-angle formulas

By setting $\phi = \theta$ in the sum formulas, we obtain the *double-angle formulas* for $\cos(2\theta)$, $\sin(2\theta)$ and $\tan(2\theta)$.

Double-angle formulas

$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

For example, using the sum formula for sine, with $\phi = \theta$, we obtain

$$\begin{aligned}\sin(2\theta) &= \sin(\theta+\theta)\\ &= \sin\theta\cos\theta + \cos\theta\sin\theta\\ &= 2\sin\theta\cos\theta,\end{aligned}$$

which is the double-angle formula for sine.

Activity 3.4 Verifying a double-angle formula

Verify the double-angle formula for the tangent function.

A solution is given on page 52.

There are two useful alternative forms of the cosine double-angle formula that can be derived using the identity $\cos^2\theta + \sin^2\theta = 1$. To obtain the first, replace $\cos^2\theta$ by $1 - \sin^2\theta$ in

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

and to obtain the second replace $\sin^2\theta$ by $1 - \cos^2\theta$.

Alternative double-angle formulas

$$\cos(2\theta) = 1 - 2\sin^2\theta$$
$$\cos(2\theta) = 2\cos^2\theta - 1$$

These alternative double-angle formulas for cosine can be rearranged to yield equations for the squares of the cosine and sine functions as follows.

Half-angle formulas

$$\sin^2\theta = \tfrac{1}{2}(1-\cos(2\theta))$$
$$\cos^2\theta = \tfrac{1}{2}(1+\cos(2\theta))$$

The name 'half-angle' refers to the fact that these formulas are often stated with θ replaced by $\frac{1}{2}\theta$ and 2θ replaced by θ. Then, for example,

$$\sin^2\left(\tfrac{1}{2}\theta\right) = \tfrac{1}{2}(1-\cos\theta).$$

By taking square roots, these formulas can be used to calculate exact values for the sine and cosine of half angles:

$$\sin\theta = \pm\sqrt{\frac{1-\cos(2\theta)}{2}} \quad \text{and} \quad \cos\theta = \pm\sqrt{\frac{1+\cos(2\theta)}{2}}.$$

For example, using the fact that $\cos(\frac{1}{4}\pi) = \frac{1}{2}\sqrt{2}$ and letting $2\theta = \frac{1}{4}\pi$, so $\theta = \frac{1}{8}\pi$, we obtain

$$\sin\left(\tfrac{1}{8}\pi\right) = \sqrt{\frac{1-\cos(\frac{1}{4}\pi)}{2}} = \sqrt{\frac{1-\frac{1}{2}\sqrt{2}}{2}} = \tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

and

$$\cos\left(\tfrac{1}{8}\pi\right) = \sqrt{\frac{1+\cos(\frac{1}{4}\pi)}{2}} = \sqrt{\frac{1+\frac{1}{2}\sqrt{2}}{2}} = \tfrac{1}{2}\sqrt{2+\sqrt{2}}.$$

In these cases, the positive sign is chosen because $0 < \frac{1}{8}\pi < \frac{1}{2}\pi$.

Activity 3.5 Using the half-angle formulas

Find $\cos\theta$ and $\sin\theta$, where θ is the angle in the interval $(0, \frac{1}{2}\pi)$ for which $\cos(2\theta) = -\frac{3}{8}$.

A solution is given on page 52.

The scaling and translation of graphs of real functions was discussed in MST121 Chapter A3, Section 2.

Each of the half-angle formulas may be interpreted in terms of two scalings and a vertical translation. This interpretation shows the remarkable fact that the graphs of the functions $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$ are both sinusoidal. For example, to obtain the graph of $g(x) = \cos^2 x$, which is the curve $y = \cos^2 x$, from the curve $y = \cos x$:

- ◇ first scale the curve $y = \cos x$ in the x-direction by the factor $\frac{1}{2}$, giving the curve $y = \cos(2x)$;
- ◇ then translate the curve $y = \cos(2x)$ one unit upwards, giving the curve $y = 1 + \cos(2x)$;
- ◇ finally scale the curve $y = 1 + \cos(2x)$ in the y-direction by the factor $\frac{1}{2}$, giving the curve $y = \frac{1}{2}(1 + \cos(2x))$.

The resulting curve is shown in Figure 3.5.

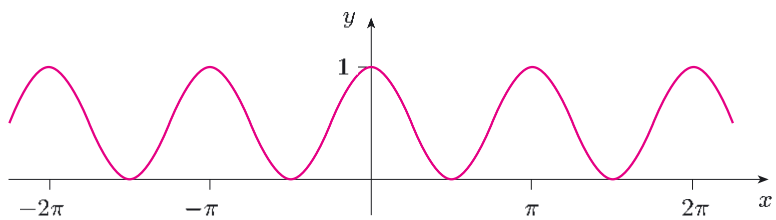


Figure 3.5 The curve $y = \cos^2 x$

Summary of Section 3

This section has reviewed or introduced:

◇ the basic relationships between the trigonometric ratios, including three versions of the Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta;$$

◇ the sum and difference formulas

$$\sin(\phi + \theta) = \sin\phi\cos\theta + \cos\phi\sin\theta$$
$$\sin(\phi - \theta) = \sin\phi\cos\theta - \cos\phi\sin\theta$$

$$\cos(\phi + \theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$
$$\cos(\phi - \theta) = \cos\phi\cos\theta + \sin\phi\sin\theta$$

$$\tan(\phi + \theta) = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta}$$
$$\tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta};$$

◇ the double-angle formulas

$$\begin{aligned}\sin(2\theta) &= 2\sin\theta\cos\theta\\ \cos(2\theta) &= \cos^2\theta - \sin^2\theta\\ &= 1 - 2\sin^2\theta\\ &= 2\cos^2\theta - 1\\ \tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta};\end{aligned}$$

◇ the half-angle formulas

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos(2\theta))$$
$$\cos^2\theta = \tfrac{1}{2}(1 + \cos(2\theta));$$

◇ ways of finding (new) values of trigonometric functions from known ones by using the above formulas.

Exercises for Section 3

Exercise 3.1

Use the exact values of sine, cosine and tangent of the angles $\frac{3}{4}\pi$ and $\frac{1}{6}\pi$ to determine exact values for the sine, cosine and tangent of $\frac{11}{12}\pi$.

Exercise 3.2

Find $\cos\theta$ and $\sin\theta$, where θ is the angle in the interval $\left(0, \frac{1}{2}\pi\right)$ for which $\cos(2\theta) = -\frac{1}{9}$.

Exercise 3.3

Find $\cos\theta$, where θ is the angle in the interval $(\frac{1}{2}\pi, \pi)$ such that $\cot\theta = -\frac{12}{5}$.

4 Quadratic curves revisited

Quadratic curves were defined in Chapter A2, Section 4.

In this section, we continue the investigation of the shape of quadratic curves begun in the previous chapter. There we concentrated on quadratics without an xy-term – that is, curves with an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

in which $B = 0$. In the absence of the xy-term, we can usually complete squares, as in equation (1.2) for the ellipse E, and so recognise the curve as an *ellipse*, a *hyperbola* or a *parabola*. In exceptional cases, a quadratic curve may be *degenerate* (e.g. a pair of lines, a single line, a point, or the empty set).

For example, the curve with equation $x^2 + y^2 + 1 = 0$ is the empty set.

But what happens if a quadratic curve has an equation which includes an xy-term? Does it still represent a conic, and if so, what type? We tackle these questions by exploring whether it is possible to find an isometry that maps such a quadratic curve onto a curve which we *know* to be a conic. The algebraic manipulations needed in this section are at times quite involved. Using these manipulations, however, we are able to develop a strategy for sketching quadratic curves with an xy-term.

René Descartes (1596–1650) discussed these questions in his book *La géometrie*. He gave few algebraic details in order not to deprive the reader of the pleasure of discovery! The first clear account was given by Fermat in his book *Introduction to plane and solid loci*.

4.1 An illustrative example

To see what can happen when an xy-term is present, consider the quadratic curve with the simple equation

$$xy - 1 = 0.$$

One way to sketch this curve is to observe that for any point (x, y) on the curve, we have $y = 1/x$. (No point (x, y) on the curve $xy - 1 = 0$ has $x = 0$ or $y = 0$.) So all we have to do is sketch the graph of the reciprocal function $f(x) = 1/x$, as shown in Figure 4.1.

The graph of the reciprocal function is discussed in MST121, Chapter A3, Section 1.

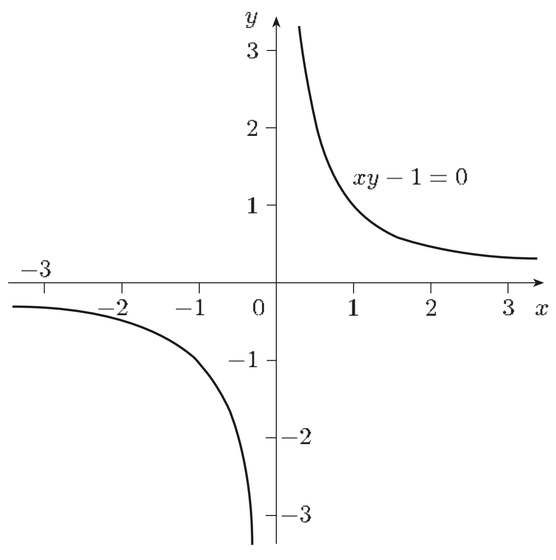


Figure 4.1 The curve $xy - 1 = 0$

Activity 4.1 Is this curve a conic?

Do you think that the curve in Figure 4.1 looks like a conic? If so, suggest an isometry that might move the curve onto a conic in standard position.

Comment

The curve looks like a hyperbola whose asymptotes lie along the x- and y-axes. If it is indeed a hyperbola, then its axes of symmetry are the lines $y = x$ and $y = -x$, for the curve is symmetric about these lines. The curve certainly cannot be *translated* onto a hyperbola in standard position, since the axes of symmetry are not parallel to the x- and y-axes. However, a clockwise *rotation* about the origin through $\frac{1}{4}\pi$ radians might work.

To test whether the suggestion in the comment on Activity 4.1 is correct, we first rotate the curve $xy - 1 = 0$ clockwise about the origin through $\frac{1}{4}\pi$ radians to obtain a new congruent curve, K say, as shown in Figure 4.2(a). We then find the equation of this rotated curve K.

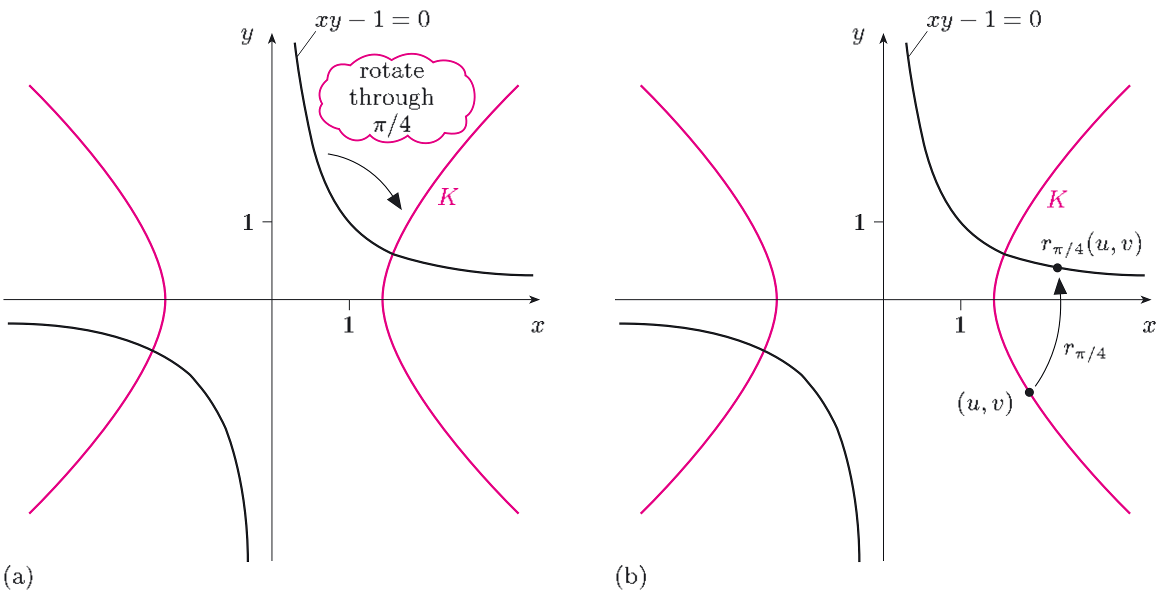


Figure 4.2 Rotating the curve $xy - 1 = 0$

Let (u, v) be an arbitrary point on the curve K. If we rotate (u, v) *anticlockwise* about the origin through $\frac{1}{4}\pi$ radians, then it ends up at the point $r_{\pi/4}(u, v)$ on the curve $xy - 1 = 0$, as shown in Figure 4.2(b). Now

$$\begin{aligned} r_{\pi/4}(u, v) &= \left(u \cos\left(\tfrac{1}{4}\pi\right) - v \sin\left(\tfrac{1}{4}\pi\right), u \sin\left(\tfrac{1}{4}\pi\right) + v \cos\left(\tfrac{1}{4}\pi\right)\right) \\ &= \left(\frac{u - v}{\sqrt{2}}, \frac{u + v}{\sqrt{2}}\right), \end{aligned}$$

See Subsection 2.3.

since $\cos(\frac{1}{4}\pi) = \sin(\frac{1}{4}\pi) = 1/\sqrt{2}$. Because the point $r_{\pi/4}(u, v)$ lies on the curve $xy - 1 = 0$, its coordinates satisfy $xy - 1 = 0$. Thus we have

$$\left(\frac{u - v}{\sqrt{2}}\right)\left(\frac{u + v}{\sqrt{2}}\right) - 1 = 0.$$

By the difference of two squares, this equation is equivalent to

$$\tfrac{1}{2}(u^2 - v^2) = \tfrac{1}{2}u^2 - \tfrac{1}{2}v^2 = 1.$$

But (u, v) is an *arbitrary* point on K, so in terms of the usual notation the equation of K is

$$\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 = 1.$$

Thus K is indeed a hyperbola in standard position, with $a = \sqrt{2}$ and $b = \sqrt{2}$. This confirms the suggestion made in the comment on Activity 4.1.

4.2 Eliminating the cross-term

In the previous subsection, you saw that, by rotating the quadratic curve $xy - 1 = 0$ clockwise about the origin through $\frac{1}{4}\pi$ radians, the hyperbola $\frac{1}{2}x^2 - \frac{1}{2}y^2 = 1$ is obtained. This rotation enabled us to eliminate the xy-term from $xy - 1 = 0$, in the sense that the equation of the rotated hyperbola has no xy-term. We now describe how the xy-term, or *cross-term*, can be eliminated from any quadratic curve L with equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \qquad (4.1)$$

Since nothing needs to be done if $B = 0$, we shall assume that $B \neq 0$.

Fermat used this technique to eliminate the xy-term.

As before, we aim to eliminate the xy-term by rotating the curve L clockwise about the origin through a suitable angle θ to obtain a new curve K which is congruent to L but whose equation has no xy-term.

Figure 4.3 shows L as an ellipse, but it could be any quadratic curve.

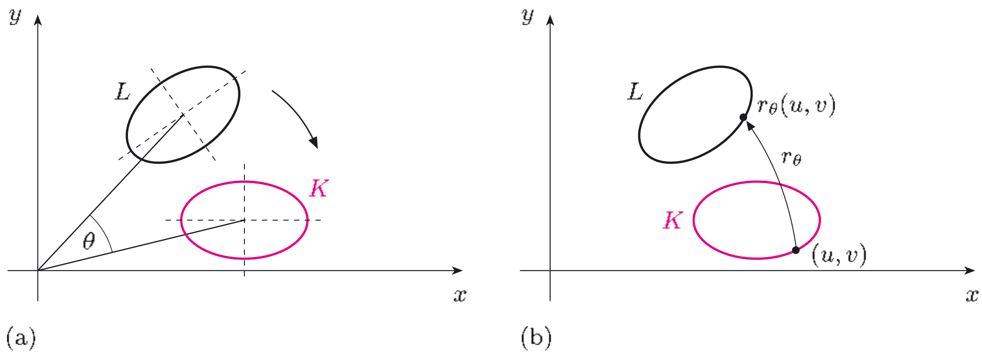


Figure 4.3 Rotating the quadratic curve L

Suppose then that K is obtained by rotating L about the origin through angle θ, as shown in Figure 4.3(a). If (u, v) is an arbitrary point on the curve K, then $r_\theta(u, v)$ lies on the curve L, as in Figure 4.3(b). Now

See Subsection 2.3.

$$r_\theta(u, v) = (u\cos\theta - v\sin\theta, u\sin\theta + v\cos\theta).$$

Because $r_\theta(u, v)$ lies on the curve L, the coordinates of $r_\theta(u, v)$,

$$x = u\cos\theta - v\sin\theta \quad \text{and} \quad y = u\sin\theta + v\cos\theta, \qquad (4.2)$$

must satisfy equation (4.1), the equation of L.

If we substitute equations (4.2) for x and y into equation (4.1), expand the brackets and collect together like terms, then, as you will see, we obtain an equation of the form

$$A'u^2 + B'uv + C'v^2 + D'u + E'v + F' = 0, \tag{4.3}$$

where the new coefficients A', B', C', D', E' and F' depend on A, B, C, D, E, F and the angle θ. If we can *choose* the angle θ so that $B' = 0$, then equation (4.3) is of the form $A'u^2 + C'v^2 + D'u + E'v + F' = 0$. Since this equation holds for *all* points (u, v) on K, the equation of K is

$$A'x^2 + C'y^2 + D'x + E'y + F' = 0, \tag{4.4}$$

and this equation has no xy-term, as we wanted.

The first stage in this process is to find the new coefficients A', B', C', D', E' and F'. You are asked to do this in the following activity.

Activity 4.2 Finding the new coefficients

By substituting equations (4.2) for x and y into equation (4.1), express the coefficients A', B', C', D', E' and F' in equation (4.3) in terms of A, B, C, D, E, F and θ.

Solutions are given on page 52.

The algebraic manipulations required here are quite involved, and you may prefer to read the solution if you are short of time.

The solution of Activity 4.2 gives the following formula for B':

$$B' = 2(C - A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta).$$

This formula may be simplified somewhat by using the double-angle formulas for sine and cosine:

See Subsection 3.3.

$$B' = (C - A)\sin(2\theta) + B\cos(2\theta).$$

Thus, in order that $B' = 0$, the angle θ must be chosen so that

$$(C - A)\sin(2\theta) + B\cos(2\theta) = 0. \tag{4.5}$$

Recall that we wish to choose θ so that $B' = 0$, because we seek a curve K whose equation has the form of equation (4.4).

Two cases arise. First, suppose that $C = A$, as in the example $xy - 1 = 0$. Then equation (4.5) is simply $B\cos(2\theta) = 0$, and this is satisfied if we choose, for example, $2\theta = \frac{1}{2}\pi$; that is, $\theta = \frac{1}{4}\pi$.

Only one value for θ is needed.

Next suppose that $C \neq A$, so $C - A \neq 0$. Then equation (4.5) gives

$$\tan(2\theta) = \frac{B}{A - C}.$$

The quantity $B/(A - C)$ may be positive or negative.

Thus a suitable value of 2θ is given by

$$2\theta = \arctan\left(\frac{B}{A - C}\right).$$

The function arctan is discussed in MST121, Chapter A3. It has image set $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$.

The corresponding value

$$\theta = \tfrac{1}{2}\arctan\left(\frac{B}{A - C}\right)$$

then lies in the interval $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$ and, for this value of θ, equation (4.5) is satisfied.

This reasoning shows that, for *any* values of the coefficients A, B and C, we can choose an angle θ, with $-\frac{1}{4}\pi < \theta \leq \frac{1}{4}\pi$, such that equation (4.5) holds and thus $B' = 0$. We call θ the **inclination** of the quadratic curve L; the reason for this name will appear shortly.

Once the inclination θ of a quadratic curve L is known, we can find the coefficients A', C', D', E' and F' in equation (4.4) by using the formulas found in Activity 4.2:

$$\begin{cases} A' = A\cos^2\theta + B\sin\theta\cos\theta + C\sin^2\theta, \\ C' = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta, \\ D' = D\cos\theta + E\sin\theta \\ E' = -D\sin\theta + E\cos\theta \\ F' = F. \end{cases} \tag{4.6}$$

We can then sketch K using the techniques of Chapter A2, and hence sketch $L = r_\theta(K)$, as in the following example in which $D = E = 0$.

Example 4.1 Sketching a quadratic curve

Sketch the quadratic curve with equation

$$19x^2 + 6xy + 11y^2 - 40 = 0.$$

Solution

First we find the inclination θ of L. Since $A = 19$, $B = 6$ and $C = 11$, we have

$$\tan(2\theta) = \frac{6}{19-11} = \frac{3}{4} = 0.75.$$

Thus the inclination is

$$\theta = \tfrac{1}{2}\arctan(0.75) \simeq 0.322 \text{ radians},$$

which is approximately 18°.

This method is discussed further after the solution.

Now, we need to find the values of $\cos^2\theta$, $\sin\theta\cos\theta$ and $\sin^2\theta$, in order to calculate A' and C' from equations (4.6). One method is to note that, since $\tan(2\theta) = \frac{3}{4}$ and 2θ lies in the first quadrant, we can use a '3-4-5 triangle' to deduce that

$$\cos(2\theta) = \tfrac{4}{5} \quad \text{and} \quad \sin(2\theta) = \tfrac{3}{5}.$$

See Subsection 3.3.

Then, by the half-angle formulas,

$$\begin{aligned} \cos^2\theta &= \tfrac{1}{2}(1+\cos(2\theta)) = \tfrac{1}{2}\left(1+\tfrac{4}{5}\right) = \tfrac{9}{10}, \\ \sin^2\theta &= \tfrac{1}{2}(1-\cos(2\theta)) = \tfrac{1}{2}\left(1-\tfrac{4}{5}\right) = \tfrac{1}{10}, \end{aligned}$$

and also

$$\sin\theta\cos\theta = \tfrac{1}{2}\sin(2\theta) = \tfrac{1}{2} \times \tfrac{3}{5} = \tfrac{3}{10}.$$

So, from equations (4.6),

$$\begin{aligned} A' &= 19 \times \tfrac{9}{10} + 6 \times \tfrac{3}{10} + 11 \times \tfrac{1}{10} = 20, \\ C' &= 19 \times \tfrac{1}{10} - 6 \times \tfrac{3}{10} + 11 \times \tfrac{9}{10} = 10. \end{aligned}$$

Next, we have $D' = 0$ and $E' = 0$, since $D = 0$ and $E = 0$, and also $F' = F = -40$. Thus equation (4.4), the equation of K, is

$$20x^2 + 10y^2 - 40 = 0; \quad \text{that is,} \quad \tfrac{1}{2}x^2 + \tfrac{1}{4}y^2 = 1.$$

Therefore K is an ellipse, as shown in Figure 4.4.

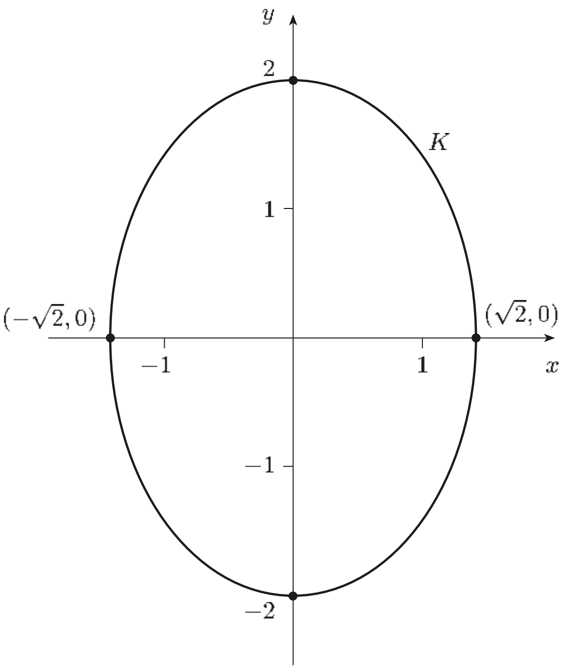


Figure 4.4 The ellipse K

The equation of K has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with $a = \sqrt{2}$ and $b = 2$. Since $a < b$, the ellipse K is *not* in standard position, but this does not matter for our current purpose.

The original curve L is obtained by rotating the ellipse K through the angle of inclination $\theta \simeq 18°$ anticlockwise about the origin. Thus L is also an ellipse, as shown in Figure 4.5.

The two broken lines are the axes of symmetry of the ellipse L.

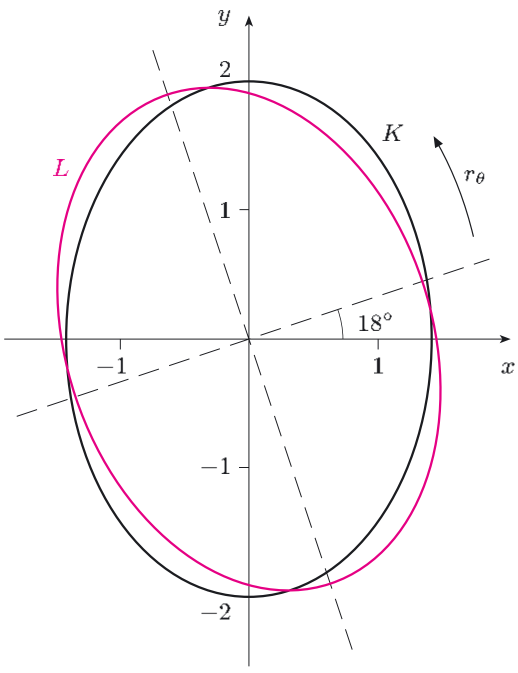


Figure 4.5 The quadratic curve $L = r_\theta(K)$

We make some remarks about the techniques used to solve problems of this type.

1. The method used in Example 4.1 for finding the values of $\cos^2\theta$, $\sin\theta\cos\theta$ and $\sin^2\theta$ from the value of $\tan(2\theta)$ avoids the inaccuracies which might occur if we use an approximate value for θ.
2. If θ lies in the interval $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$, then we can find $\cos(2\theta)$ and $\sin(2\theta)$ from $\tan(2\theta)$ either by using an appropriate right-angled triangle (as in Example 4.1) or by using the formulas

$$\cos(2\theta) = \frac{1}{\sqrt{1+\tan^2(2\theta)}}, \quad \sin(2\theta) = \frac{\tan(2\theta)}{\sqrt{1+\tan^2(2\theta)}}. \tag{4.7}$$

These formulas are obtained from equations (3.3), on replacing θ by 2θ. We can take the + sign in both cases here because, for θ in $(-\frac{1}{4}\pi, \frac{1}{4}\pi)$,

See Figure 3.2.

◇ $\cos(2\theta)$ is positive;

◇ $\sin(2\theta)$ and $\tan(2\theta)$ have the same sign.

3. In cases like $xy - 1 = 0$, discussed in Subsection 4.1, where $\theta = \frac{1}{4}\pi$, we have $\cos\theta = \sin\theta = \frac{1}{2}\sqrt{2}$. So

$$\cos^2\theta = \sin^2\theta = \sin\theta\cos\theta = \tfrac{1}{2}.$$

4. If $D = E = 0$, as in Example 4.1, then $D' = E' = 0$. In this case K is a curve with equation of the form $A'x^2 + C'y^2 + F' = 0$ and so is usually an ellipse or hyperbola. Such an ellipse or hyperbola must either be in standard position or it can be obtained from an ellipse or hyperbola in standard position by exchanging the roles of the x- and y-coordinates, that is, by reflection in the line $y = x$.

See Chapter A2, Activity 2.1(b).

An ellipse in *reflected standard position* has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } b > a > 0).$$

A hyperbola in *reflected standard position* has an equation of the form

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } a > 0 \text{ and } b > 0).$$

Figure 4.6 illustrates an ellipse and a hyperbola in reflected standard position, showing their vertices and the asymptotes of the hyperbola.

For example, the equation

$$x^2/9 + y^2/16 = 1$$

represents an ellipse in reflected standard position and the equation

$$-x^2/10 + y^2/3 = 1$$

represents a hyperbola in reflected standard position.

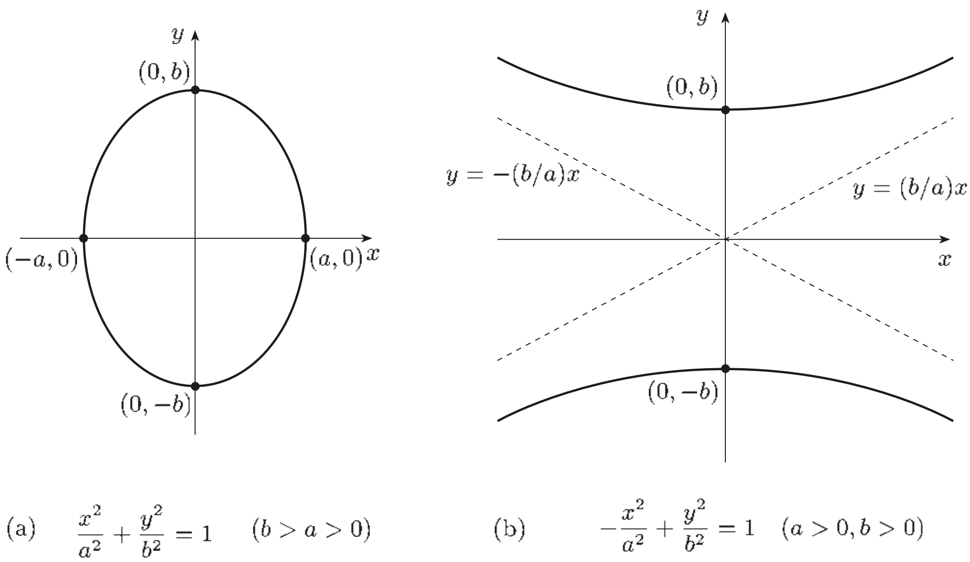


Figure 4.6 Ellipse and hyperbola in reflected standard position

Therefore a quadratic curve L with equation of the form $Ax^2 + Bxy + Cy^2 + F = 0$, where $B \neq 0$, is usually an ellipse or hyperbola, with one axis of symmetry inclined at an angle θ, where $-\frac{1}{4}\pi < \theta \leq \frac{1}{4}\pi$, to the positive x-axis; this explains why θ is called the inclination of L.

5. The curve L is obtained by rotating the ellipse or hyperbola K through the angle θ (anticlockwise if $\theta > 0$ and clockwise if $\theta < 0$). Thus, in order to sketch L, you need to be able to draw an angle of θ, at least approximately. To do this, if no protractor is available, it is sufficient to estimate what fraction of a right angle θ is. Using your estimate, you can draw the axes of symmetry of L, one of which is inclined at an angle θ to the positive x-axis (as in Figure 4.5). Finally, sketch L by rotating your sketch of K through angle θ and copying it (by eye or by tracing it).

Here is a summary of the method used in Example 4.1.

Strategy

To sketch a quadratic curve L with equation

$$Ax^2 + Bxy + Cy^2 + F = 0 \quad (\text{where } B \neq 0).$$

1. Find the inclination θ of L, given by
$$\theta = \begin{cases} \frac{1}{4}\pi, & \text{if } A = C, \\ \frac{1}{2}\arctan\left(\dfrac{B}{A-C}\right), & \text{if } A \neq C. \end{cases}$$
2. Obtain the coefficients of the equation $A'x^2 + C'y^2 + F' = 0$ of K, using equations (4.6).
3. Sketch the conic K.
4. Sketch $L = r_\theta(K)$ by rotating the sketch of K through θ.

In this strategy, $D = 0$ and $E = 0$, as in Example 4.1.

You are now asked to use this strategy to sketch a quadratic curve.

Activity 4.3 Sketching a quadratic curve

Let L be the quadratic curve with equation

$$x^2 + 12xy + 6y^2 - 30 = 0.$$

Use the above strategy to sketch L.

A solution is given on page 52.

The strategy enables us to sketch any conic L with equation of the form $Ax^2 + Bxy + Cy^2 + F = 0$ as the rotation of an ellipse or hyperbola K in standard position or reflected standard position. In Section 5 we plot such quadratic curves L by rotating the known parametrisations for K.

Finally, we briefly mention quadratic curves with equations like

$$9x^2 - 24xy + 16y^2 - 10x - 70y - 75 = 0,$$

which have a non-zero xy-term and also non-zero x and/or y terms. The technique based on equation (4.5) for eliminating the xy-term by rotation works equally well in this case. The only difference is that elimination of the xy-term results in a congruent curve K whose equation contains

non-zero x and/or y terms. But, as described at the beginning of this section, such curves K can usually be recognised as ellipses, hyperbolas or parabolas.

In summary, all quadratic curves, except for certain degenerate cases, are ellipses, hyperbolas or parabolas. This answers the question posed about quadratic curves at the beginning of Chapter A2, Section 4.

Summary of Section 4

This section has introduced:

- ◇ the inclination θ of a quadratic curve;
- ◇ a strategy for removing the cross-term from the equation of a quadratic curve by rotating the curve through the angle θ;
- ◇ the idea that every quadratic curve is, except for degenerate cases, an ellipse, hyperbola or parabola..

Exercises for Section 4

Exercise 4.1

Use the strategy to sketch the quadratic curve L with each of the following equations.

(a) $8x^2 + 4xy + 5y^2 - 36 = 0$

(b) $x^2 - 10xy + y^2 + 8 = 0$

5 Isometries on the computer

To study this section you will need access to your computer, together with the Mathcad files and Computer Book for Block A.

This section shows how the computer can be used to translate, rotate and reflect triangles and conics.

Refer to Computer Book A for the work in this subsection.

Summary of Section 5

After studying this section, you should be able to use the computer to:

◇ draw triangles;

◇ define translations, rotations about the origin, and reflections in lines through the origin, and apply them to triangles and conics.

Summary of Chapter A3

In this chapter the idea of a function has been developed by using functions to describe various geometrical phenomena in the plane $\mathbb{R}^2$. In particular we have introduced parametrisation functions, which describe curves in $\mathbb{R}^2$, and we have introduced isometries, which describe rigid-body motions of figures in $\mathbb{R}^2$. The latter enable us to sketch non-degenerate conics by expressing them as images of a conic in standard position or reflected standard position under an isometry.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Function, domain, codomain, rule, image, image set, parametrisation function, function of two variables, surface and contour plots, isometry, translation, rotation and reflection functions, self-inverse, glide-reflection, composite isometry, the trigonometric sum/ difference/ double- and half-angle formulas, the Pythagorean identity, cross-term, inclination of a quadratic curve, reflected standard position.

Symbols and notation to know and use

- ◇ The two-line notation for functions.
- ◇ The image $f(x)$ of an element x, and the image $f(S)$ of a set S.
- ◇ $\mathbb{R}^2$ for the Cartesian plane.
- ◇ r_θ for rotation, q_θ for reflection, and $t_{p,q}$ for translation.
- ◇ $\circ$ for composition of functions.

Mathematical skills

- ◇ Write down a parametrisation function for a conic in standard position.
- ◇ Write down an algebraic description for a function that represents a translation, a rotation about the origin or a reflection in a line through the origin.
- ◇ Determine the composite of two simple isometries.
- ◇ Give a geometrical interpretation for an isometry that has been expressed algebraically.
- ◇ Express (up to a choice of $+$ or $-$ sign) any one of the trigonometric ratios $\sin\theta$, $\cos\theta$, $\tan\theta$, $\operatorname{cosec}\theta$, $\sec\theta$, $\cot\theta$ in terms of any other. Determine the sign by reference to the quadrant in which θ lies.
- ◇ Use the sum, difference, double-angle, or half-angle formulas, to obtain new exact values of trigonometric functions from existing exact values.
- ◇ Find the inclination of a quadratic curve.
- ◇ Sketch a quadratic curve of the form $Ax^2 + Bxy + Cy^2 + F = 0$ by eliminating the xy-term.

Mathcad skills

- ◇ Plot triangles.
- ◇ Apply translations, rotations about the origin, and reflections in lines through the origin, to triangles and conics.

Ideas to be aware of

- ◇ Curves (including non-degenerate conics) may be represented by parametrisation functions.
- ◇ Every isometry of $\mathbb{R}^2$ is a translation, rotation, reflection, or glide-reflection.
- ◇ Every quadratic curve is, except for degenerate cases, an ellipse, hyperbola or parabola.

Summary of Block A

This block has introduced three key topics:

- ◇ linear second-order recurrence sequences;
- ◇ conics;
- ◇ functions whose domain and/or codomain is $\mathbb{R}^2$.

You saw in Chapter A1 that there is a strategy for finding closed forms of many linear second-order recurrence sequences, including the Fibonacci sequence, and also that patterns can often be found in the terms of these sequences.

In Chapter A2 you saw how to sketch those non-degenerate conics which are either in standard position or are translations of conics in standard position.

In this chapter you met isometries, including rotations, reflections and translations, and saw that other conics can be sketched by using rotations, reflections and translations of conics in standard position.

Amongst the topics in Block B, you will learn more about recurrence sequences. You will also meet a new method of representing certain types of functions, including isometries, from $\mathbb{R}^2$ to $\mathbb{R}^2$. This new method uses a mathematical object called a *matrix*, which is a rectangular array of numbers. Matrices can be combined using operations such as addition and multiplication which are similar, but not identical, to the corresponding operations on numbers. The use of matrix multiplication will help us to understand the long-term behaviour of certain recurrence sequences.

Solutions to Activities

Solution 1.1

The point $(5,1)$ lies on E because

$$\tfrac{1}{9}(5-2)^2 + \tfrac{1}{4}(1-1)^2 = \tfrac{1}{9}(3)^2 + \tfrac{1}{4}(0)^2 = 1.$$

If $(5,1)$ is translated two units to the left and one unit down it ends up at the point $(5-2, 1-1) = (3,0)$. This point lies on the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ because

$$\tfrac{1}{9}(3)^2 + \tfrac{1}{4}(0)^2 = 1.$$

Solution 1.2

In this case $t(x,y) = (x-2, y-1)$, so

$$\begin{aligned} t(1,-3) &= (1-2, -3-1) = (-1,-4), \\ t(2,7) &= (2-2, 7-1) = (0,6), \\ t(-2,4) &= (-2-2, 4-1) = (-4,3). \end{aligned}$$

Solution 1.3

The function that maps the ellipse $\frac{1}{9}x^2 + \frac{1}{4}y^2 = 1$ back onto E must translate points two units to the right and one unit up. In other words, t is the function defined by

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+2, y+1). \end{aligned}$$

Solution 1.4

Here $p(t) = (2 + 3\cos t, 1 + 2\sin t)$, so

$$\begin{aligned} p(0) &= (2 + 3\cos(0), 1 + 2\sin(0)) = (5,1), \\ p(\pi) &= (2 + 3\cos(\pi), 1 + 2\sin(\pi)) = (-1,1), \\ p\left(\tfrac{4}{3}\pi\right) &= \left(2 + 3\cos\left(\tfrac{4}{3}\pi\right), 1 + 2\sin\left(\tfrac{4}{3}\pi\right)\right) \\ &= \left(2 - \tfrac{3}{2}, 1 - \sqrt{3}\right) \simeq (0.5, -0.73). \end{aligned}$$

These are the points of E shown below.

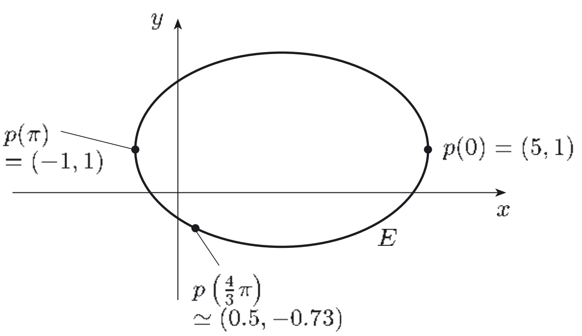


Figure S.1

As t ranges through the interval $[0, 2\pi]$ the image $p(t)$ traces out the ellipse E, so the image set of p is E.

Solution 1.5

(a) The circle has radius 2 and centre $(2,5)$, so it has parametric equations

$$x = 2 + 2\cos t, \quad y = 5 + 2\sin t \quad (0 \le t \le 2\pi).$$

Thus the circle has parametrisation function

$$\begin{aligned} p : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (2 + 2\cos t, 5 + 2\sin t). \end{aligned}$$

(b) This is a parabola in standard position

$$y^2 = 4ax,$$

with $a = 2$, so it has parametric equations

$$x = 2t^2, \quad y = 4t.$$

Thus the parabola has parametrisation function

$$\begin{aligned} p : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (2t^2, 4t). \end{aligned}$$

(c) This is a hyperbola in standard position

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

with $a = \sqrt{3}$ and $b = \sqrt{10}$, so it has parametric equations

$$x = \sqrt{3}\sec t, \quad y = \sqrt{10}\tan t$$
$$\left(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi\right).$$

A parametrisation function for the right branch is thus

$$\begin{aligned} p : \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right) &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (\sqrt{3}\sec t, \sqrt{10}\tan t). \end{aligned}$$

Solution 2.1

(a) The required translation is

$$\begin{aligned} t_{3,-2} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (x+3, y-2). \end{aligned}$$

(b) Under this translation the points $(2,1)$, $(5,3)$, $(3,4)$ map to the points $(5,-1)$, $(8,1)$, $(6,2)$, respectively. It follows that the triangle T maps to the triangle T' with vertices at $(5,-1)$, $(8,1)$ and $(6,2)$, as shown below.

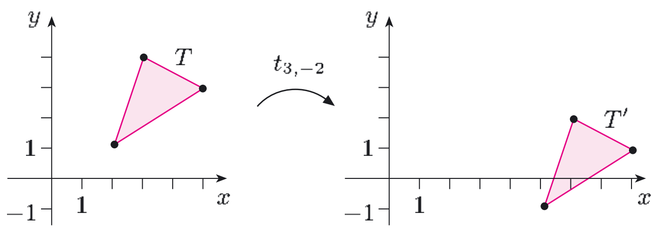


Figure S.2

Solution 2.2

(a) The inverse of the translation $t_{3,-2}$ in Activity 2.1 is

$$\begin{aligned} t_{-3,2} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x-3, y+2). \end{aligned}$$

(b) Under this inverse the vertices $(5,-1)$, $(8,1)$, $(6,2)$ of triangle T' map back to the vertices $(2,1)$, $(5,3)$, $(3,4)$ of T, so $t_{-3,2}(T') = T$, as expected.

Solution 2.3

The following figure illustrates the effect that the translations have on a triangle. First $t_{3,6}$ moves the triangle three units to the right and six units up, then $t_{1,-2}$ shifts the triangle a further one unit to the right and two units down.

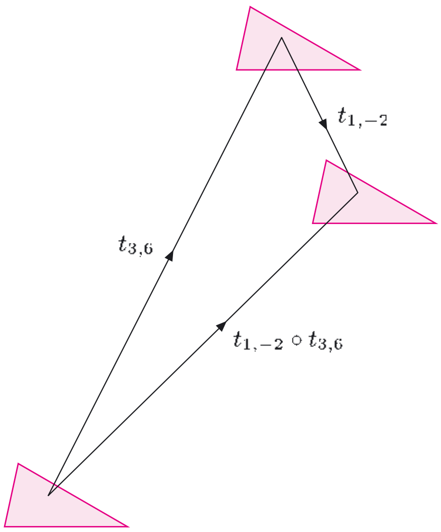


Figure S.3

Overall the composite $t_{1,-2} \circ t_{3,6}$ shifts the original triangle four units to the right and four units up. In other words,

$$t_{1,-2} \circ t_{3,6} = t_{4,4}.$$

Solution 2.4

(a) Prior to rotation the triangle T has a side of length 3 along the x-axis and a side of length 2 parallel to the y-axis. After rotation the triangle ends up in the second quadrant with the side of length 3 along the y-axis and the side of length 2 parallel to the x-axis, as shown below.

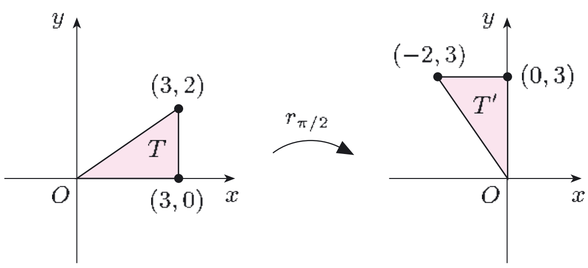


Figure S.4

(b) It follows that the point $(3,2)$ rotates to the point $(-2,3)$.

Solution 2.5

The coordinates of the images of the points under $r_{\pi/2}$ are as follows:

$$\begin{aligned} r_{\pi/2}(-3,2) &= (-2,-3), \\ r_{\pi/2}(1,0) &= (0,1), \\ r_{\pi/2}(0,1) &= (-1,0), \\ r_{\pi/2}(-2,-3) &= (3,-2). \end{aligned}$$

The effect of $r_{\pi/2}$ on the points is shown below.

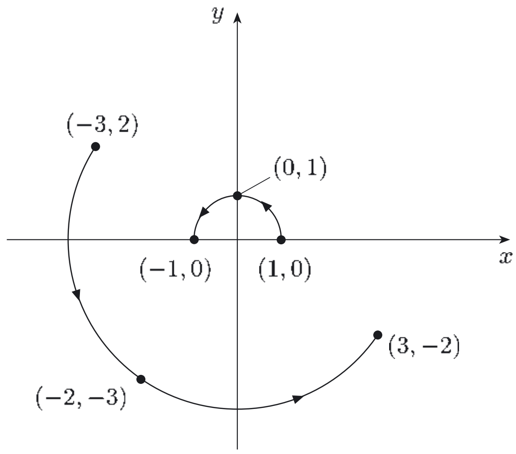


Figure S.5

Solution 2.6

Consider the projection of the sides OQ' and $Q'P'$ onto the y-axis, as indicated by the dotted lines in the figure below.

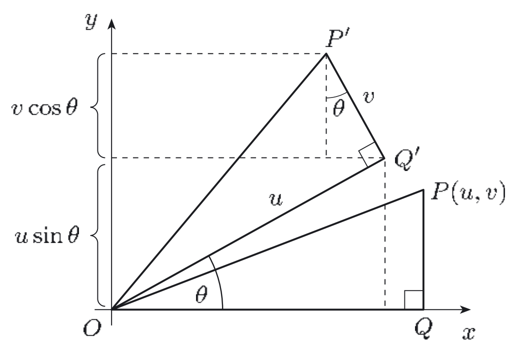


Figure S.6

From the figure, observe that the projection of $Q'P'$ has length $v\cos\theta$, and the projection of OQ' has length $u\sin\theta$. The y-coordinate of the point P' is therefore $u\sin\theta + v\cos\theta$.

Solution 2.7

(a) Substituting the values

$$\cos\pi = -1 \quad \text{and} \quad \sin\pi = 0$$

into the general equation for a rotation about the origin, we obtain

$$r_\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (-x,-y).$$

Under this rotation the point $(1,0)$ is sent to the point $(-1,0)$, which is what you would expect from half a turn about the origin anticlockwise.

(b) Substituting the values

$$\cos\left(\tfrac{1}{4}\pi\right) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin\left(\tfrac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}$$

into the general equation for a rotation about the origin we obtain

$$r_{\pi/4} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right).$$

Under this rotation the point $(1,0)$ is sent to the point $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$, which is what you would expect from an eighth of a turn about the origin anticlockwise.

(c) Substituting the values

$$\cos\left(-\tfrac{2}{3}\pi\right) = -\tfrac{1}{2} \quad \text{and} \quad \sin\left(-\tfrac{2}{3}\pi\right) = -\tfrac{1}{2}\sqrt{3}$$

into the general equation for a rotation about the origin, we obtain

$$r_{-2\pi/3} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto \left(-\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{3}y, -\tfrac{1}{2}\sqrt{3}x - \tfrac{1}{2}y\right).$$

Under this rotation the point $(1,0)$ is sent to the point $\left(-\tfrac{1}{2}, -\tfrac{1}{2}\sqrt{3}\right)$, which is what you would expect from one third of a turn about the origin clockwise.

Solution 2.8

(a) The composite is a rotation through the angle $\frac{1}{2}\pi + \frac{1}{3}\pi = \frac{5}{6}\pi$. Since this angle lies in the interval $(-\pi, \pi]$, the required answer is $r_{5\pi/6}$.

(b) Here the composite is a rotation through the angle $\frac{2}{3}\pi + \frac{1}{2}\pi = \frac{7}{6}\pi$. To bring this angle into the interval $(-\pi, \pi]$ we subtract 2π. The required answer is therefore $r_{-5\pi/6}$.

(c) In this case the composite is a rotation through the angle $\frac{1}{3}\pi - \frac{1}{2}\pi = -\frac{1}{6}\pi$. Since this angle lies in the interval $(-\pi, \pi]$, the required answer is $r_{-\pi/6}$.

Solution 2.9

(a) A reflection in the x-axis fixes $(1,0)$ and $(0,0)$, it sends $(0,1)$ to $(0,-1)$, and it sends $(1,1)$ to $(1,-1)$.

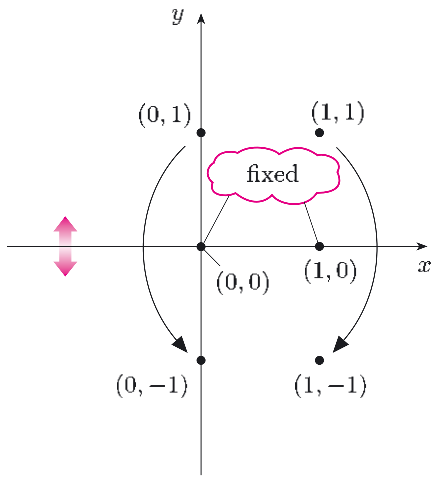


Figure S.7

In general, the reflection acts on each point by reversing the sign of the y-coordinate and keeping the x-coordinate fixed. We can therefore write the reflection in the x-axis in the form of a function as follows:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (x,-y).$$

(b) Reflection in the line $y = x$ swaps $(1,0)$ and $(0,1)$ and leaves $(0,0)$ and $(1,1)$ fixed.

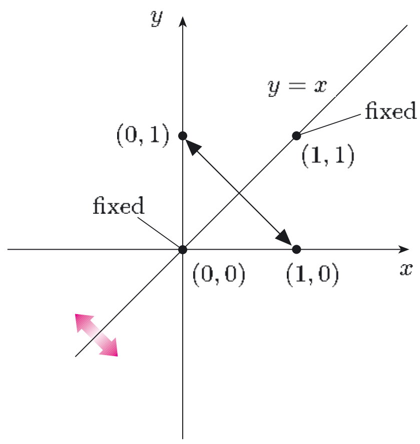


Figure S.8

In general, the reflection acts on each point by swapping its x- and y-coordinates. We can therefore write the reflection in the line $y = x$ in the form of a function as follows:

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(x,y) \longmapsto (y,x).$$

(c) Reflection in the line $y = -x$ sends $(1, 0)$ to $(0, -1)$, it sends $(0, 1)$ to $(-1, 0)$ and it sends $(1, 1)$ to $(-1, -1)$. The point $(0, 0)$ remains fixed.

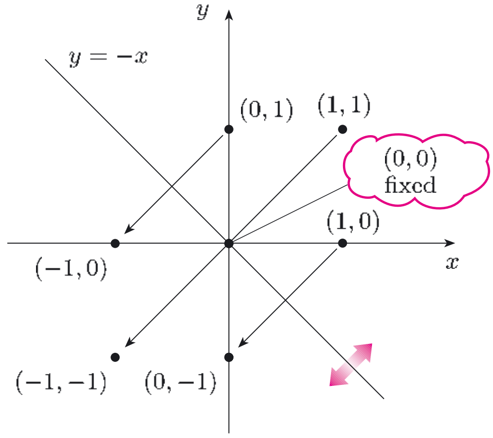


Figure S.9

In general, the reflection acts on each point by swapping the x- and y-coordinates and reversing their signs. We can therefore write the reflection in the line $y = -x$ as follows:

$$\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-y, -x). \end{aligned}$$

Solution 2.10

Each leaf motif can be sent onto its right-hand neighbour by first reflecting it in a horizontal line and then translating the reflection to the right, as shown below. (Alternatively, the reflection may be preceded by the translation.)

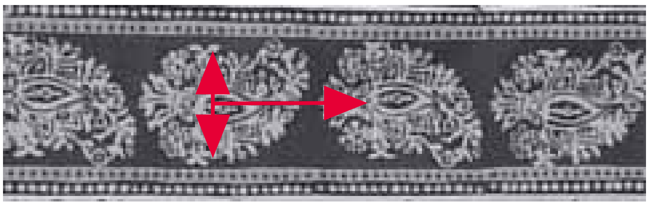


Figure S.10

Solution 2.11

For $r \circ q$ we have

$$r\,(q(x, y)) = r(x, -y) = (-y, -x),$$

so the required function is

$$\begin{aligned} r \circ q : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (-y, -x). \end{aligned}$$

As you saw in Solution 2.9(c), this is a reflection in the line $y = -x$.

For $q \circ r$ we have

$$q\,(r(x, y)) = q(y, -x) = (y, x),$$

so in this case the function is

$$\begin{aligned} q \circ r : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (y, x). \end{aligned}$$

As you saw in Solution 2.9(b), this is a reflection in the line $y = x$.

Clearly $r \circ q$ is not the same as $q \circ r$. For example, $r \circ q(1, 0) = (0, -1)$, whereas $q \circ r(1, 0) = (0, 1)$.

Solution 2.12

For $\theta = \frac{1}{6}\pi$ (i.e. 30 degrees), we have

$$\cos(2\theta) = \tfrac{1}{2} \quad \text{and} \quad \sin(2\theta) = \tfrac{1}{2}\sqrt{3}.$$

The function for the reflection in ℓ is therefore

$$\begin{aligned} q_{\pi/6} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \left(\tfrac{1}{2}x + \tfrac{1}{2}\sqrt{3}y, \tfrac{1}{2}\sqrt{3}x - \tfrac{1}{2}y\right). \end{aligned}$$

Solution 3.1

(a) By equations (3.2),

$$\cos\theta = \pm\sqrt{1 - (\tfrac{1}{2})^2} = \pm\tfrac{1}{2}\sqrt{3}$$

and

$$\tan\theta = \pm\frac{\frac{1}{2}}{\sqrt{1 - (\frac{1}{2})^2}} = \pm\frac{1}{\sqrt{3}}.$$

Since $\frac{1}{2}\pi < \theta < \pi$, $\cos\theta$ and $\tan\theta$ are negative, so

$$\cos\theta = -\tfrac{1}{2}\sqrt{3}, \quad \tan\theta = -1/\sqrt{3}.$$

(Alternatively, you could use the fact that $\theta = 5\pi/6$!)

(b) By equations (3.3),

$$\cos\theta = \pm\frac{1}{\sqrt{1 + (-\frac{3}{4})^2}} = \pm\tfrac{4}{5}$$

and

$$\sin\theta = \pm\frac{-\frac{3}{4}}{\sqrt{1 + (-\frac{3}{4})^2}} = \pm\tfrac{3}{5}.$$

Since $-\frac{1}{2}\pi < \theta < 0$, $\cos\theta > 0$ and $\sin\theta < 0$, so

$$\cos\theta = \tfrac{4}{5}, \quad \sin\theta = -\tfrac{3}{5}.$$

(c) Since θ lies in interval $(0, \frac{1}{2}\pi)$, we can make use of the triangle in the figure. We obtain

$$\sin\theta = \tfrac{1}{7}\sqrt{13} \quad \text{and} \quad \tan\theta = \tfrac{1}{6}\sqrt{13}.$$

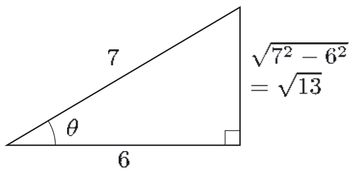


Figure S.11

Solution 3.2

We have

$$\begin{aligned}\sin\left(\tfrac{7}{12}\pi\right) &= \sin\left(\tfrac{1}{4}\pi + \tfrac{1}{3}\pi\right)\\ &= \sin\left(\tfrac{1}{4}\pi\right)\cos\left(\tfrac{1}{3}\pi\right) + \cos\left(\tfrac{1}{4}\pi\right)\sin\left(\tfrac{1}{3}\pi\right)\\ &= \tfrac{1}{2}\sqrt{2}\times\tfrac{1}{2} + \tfrac{1}{2}\sqrt{2}\times\tfrac{1}{2}\sqrt{3}\\ &= \tfrac{1}{4}\left(\sqrt{2}+\sqrt{6}\right),\end{aligned}$$

and

$$\begin{aligned}\tan\left(\tfrac{7}{12}\pi\right) &= \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{3}\pi\right)\\ &= \frac{\tan\left(\tfrac{1}{4}\pi\right) + \tan\left(\tfrac{1}{3}\pi\right)}{1-\tan\left(\tfrac{1}{4}\pi\right)\tan\left(\tfrac{1}{3}\pi\right)}\\ &= \frac{1+\sqrt{3}}{1-1\times\sqrt{3}} = -2-\sqrt{3},\end{aligned}$$

after rationalising the denominator.

Solution 3.3

We have $\frac{1}{12}\pi = \frac{1}{3}\pi - \frac{1}{4}\pi$, so

$$\begin{aligned}\sin\left(\tfrac{1}{12}\pi\right) &= \sin\left(\tfrac{1}{3}\pi - \tfrac{1}{4}\pi\right)\\ &= \sin\left(\tfrac{1}{3}\pi\right)\cos\left(\tfrac{1}{4}\pi\right) - \cos\left(\tfrac{1}{3}\pi\right)\sin\left(\tfrac{1}{4}\pi\right)\\ &= \tfrac{1}{2}\sqrt{3}\times\tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\times\tfrac{1}{2}\sqrt{2}\\ &= \tfrac{1}{4}\left(\sqrt{6}-\sqrt{2}\right),\end{aligned}$$

$$\begin{aligned}\cos\left(\tfrac{1}{12}\pi\right) &= \cos\left(\tfrac{1}{3}\pi - \tfrac{1}{4}\pi\right)\\ &= \cos\left(\tfrac{1}{3}\pi\right)\cos\left(\tfrac{1}{4}\pi\right) + \sin\left(\tfrac{1}{3}\pi\right)\sin\left(\tfrac{1}{4}\pi\right)\\ &= \tfrac{1}{2}\times\tfrac{1}{2}\sqrt{2} + \tfrac{1}{2}\sqrt{3}\times\tfrac{1}{2}\sqrt{2}\\ &= \tfrac{1}{4}\left(\sqrt{2}+\sqrt{6}\right),\end{aligned}$$

and

$$\begin{aligned}\tan\left(\tfrac{1}{12}\pi\right) &= \tan\left(\tfrac{1}{3}\pi - \tfrac{1}{4}\pi\right)\\ &= \frac{\tan(\tfrac{1}{3}\pi) - \tan(\tfrac{1}{4}\pi)}{1+\tan(\tfrac{1}{3}\pi)\tan(\tfrac{1}{4}\pi)}\\ &= \frac{\sqrt{3}-1}{1+\sqrt{3}} = 2-\sqrt{3},\end{aligned}$$

after rationalising the denominator.

Solution 3.4

By the tangent sum formula (3.7),

$$\begin{aligned}\tan(2\theta) = \tan(\theta+\theta) &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}\\ &= \frac{2\tan\theta}{1-\tan^2\theta}.\end{aligned}$$

Solution 3.5

Since θ lies in $\left(0, \frac{1}{2}\pi\right)$, $\cos\theta$ and $\sin\theta$ are positive, so

$$\cos\theta = \sqrt{\frac{1+\cos(2\theta)}{2}} = \sqrt{\frac{1-\frac{3}{8}}{2}} = \sqrt{\frac{5}{16}} = \tfrac{1}{4}\sqrt{5},$$

and

$$\sin\theta = \sqrt{\frac{1-\cos(2\theta)}{2}} = \sqrt{\frac{1+\frac{3}{8}}{2}} = \sqrt{\frac{11}{16}} = \tfrac{1}{4}\sqrt{11}.$$

Solution 4.2

If we substitute

$$x = u\cos\theta - v\sin\theta \quad \text{and} \quad y = u\sin\theta + v\cos\theta$$

into the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

then we obtain

$$\begin{aligned}&A(u\cos\theta - v\sin\theta)^2\\ &+B(u\cos\theta - v\sin\theta)(u\sin\theta + v\cos\theta)\\ &+C(u\sin\theta + v\cos\theta)^2\\ &+D(u\cos\theta - v\sin\theta)\\ &+E(u\sin\theta + v\cos\theta)\\ &+F = 0.\end{aligned}$$

Multiplying out the brackets, we obtain

$$\begin{aligned}&A(u^2\cos^2\theta - 2uv\cos\theta\sin\theta + v^2\sin^2\theta)\\ &+B(u^2\cos\theta\sin\theta + uv(\cos^2\theta - \sin^2\theta)\\ &\qquad\qquad -v^2\sin\theta\cos\theta)\\ &+C(u^2\sin^2\theta + 2uv\sin\theta\cos\theta + v^2\cos^2\theta)\\ &+D(u\cos\theta - v\sin\theta)\\ &+E(u\sin\theta + v\cos\theta)\\ &+F = 0.\end{aligned}$$

In order to rewrite this in the form

$$A'u^2 + B'uv + C'v^2 + D'u + E'v + F' = 0$$

we collect together the coefficients of u^2, uv, v^2, etc. This gives

$$\begin{aligned}&\left(A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta\right)u^2\\ &+\left(2(C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta)\right)uv\\ &+\left(A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta\right)v^2\\ &+(D\cos\theta + E\sin\theta)\,u\\ &+(-D\sin\theta + E\cos\theta)\,v\\ &+F = 0.\end{aligned}$$

From this we can read off the new coefficients:

$$\begin{aligned}A' &= A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta,\\ B' &= 2(C-A)\sin\theta\cos\theta + B(\cos^2\theta - \sin^2\theta),\\ C' &= A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta,\\ D' &= D\cos\theta + E\sin\theta,\\ E' &= -D\sin\theta + E\cos\theta,\\ F' &= F.\end{aligned}$$

Solution 4.3

Step 1: Here $A = 1$, $B = 12$ and $C = 6$, so $B/(A-C) = 12/(-5) = -2.4$. Thus the inclination is

$$\theta = \tfrac{1}{2}\arctan(-2.4) \simeq -0.588 \text{ radians},$$

which is approximately $-34°$.

Step 2: Since $\tan(2\theta) = B/(A - C) = -12/5$, equations (4.7) give

$$\cos(2\theta) = 1/\sqrt{1 + (-\tfrac{12}{5})^2} = \tfrac{5}{13},$$

and

$$\sin(2\theta) = -\tfrac{12}{5}/\sqrt{1 + (-\tfrac{12}{5})^2} = -\tfrac{12}{13}.$$

By the half-angle formulas,

$$\cos^2\theta = \tfrac{1}{2}(1 + \tfrac{5}{13}) = \tfrac{9}{13},$$
$$\sin^2\theta = \tfrac{1}{2}(1 - \tfrac{5}{13}) = \tfrac{4}{13},$$

and

$$\sin\theta\cos\theta = \tfrac{1}{2} \times (-\tfrac{12}{13}) = -\tfrac{6}{13}.$$

Hence, by equations (4.6),

$$A' = 1 \times \tfrac{9}{13} + 12 \times (-\tfrac{6}{13}) + 6 \times \tfrac{4}{13} = -3,$$
$$C' = 1 \times \tfrac{4}{13} - 12 \times (-\tfrac{6}{13}) + 6 \times \tfrac{9}{13} = 10.$$

Also $F' = F = -30$.

Hence the equation of K is

$$-3x^2 + 10y^2 - 30 = 0;$$

that is,

$$-\frac{x^2}{10} + \frac{y^2}{3} = 1.$$

Step 3: This is a hyperbola in reflected standard position, with $a = \sqrt{10}$ and $b = \sqrt{3}$, sketched using Figure 4.6(b).

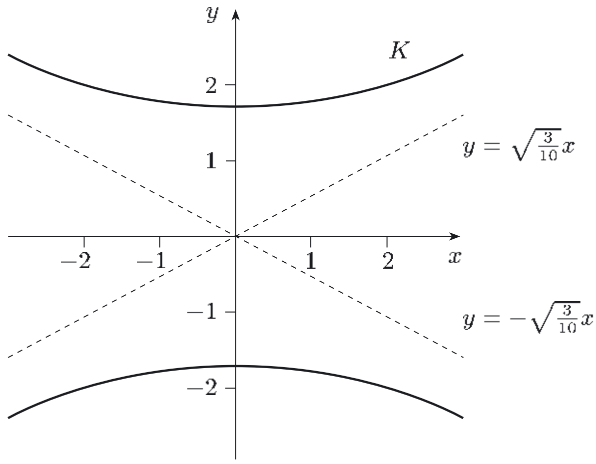


Figure S.12

Step 4: Since $L = r_\theta(K)$, where θ is approximately $-34°$, we can sketch L as follows. (Note that $34°$ is a little over one-third of a right angle.)

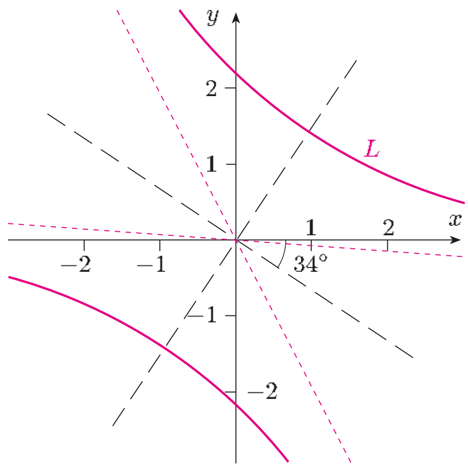


Figure S.13

In Figure S.13 the asymptotes of L are shown, but this is not essential. Note that you could have included K in Figure S.13, as was done in Figure 4.5, though this might make the figure rather confusing.

Solutions to Exercises

Solution 1.1

(a) Let us first rewrite the equation of the curve in the form

$$x^2 - y^2 - 4x + 6y - 6 = 0.$$

After completing the squares, we obtain the equivalent equation

$$(x-2)^2 - (y-3)^2 - 4 + 9 - 6 = 0;$$

that is,

$$(x-2)^2 - (y-3)^2 = 1.$$

It follows that the quadratic curve is a hyperbola that can be moved into standard position by translating two units to the left and three units down. The translation function that achieves this is

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x-2, y-3). \end{aligned}$$

The image of L under t is the hyperbola K with equation

$$x^2 - y^2 = 1.$$

(b) To map the image hyperbola K back onto the quadratic curve L we need to translate it two units to the right and three units up. The translation function that achieves this is

$$\begin{aligned} t : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x+2, y+3). \end{aligned}$$

(c) The hyperbola $x^2 - y^2 = 1$ has parametric equations

$$x = \sec t, \quad y = \tan t \quad \left(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi\right)$$

so the hyperbola $(x-2)^2 - (y-3)^2 = 1$ has parametric equations

$$x = 2 + \sec t, \quad y = 3 + \tan t \quad \left(-\tfrac{1}{2}\pi < t < \tfrac{1}{2}\pi, \tfrac{1}{2}\pi < t < \tfrac{3}{2}\pi\right).$$

A suitable parametrisation function for the right branch of L is therefore

$$\begin{aligned} p : \left(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right) &\longrightarrow \mathbb{R}^2 \\ t &\longmapsto (2 + \sec t, 3 + \tan t). \end{aligned}$$

There is a similar parametrisation function for the left branch of L, with the same rule but domain $\left(\tfrac{1}{2}\pi, \tfrac{3}{2}\pi\right)$.

Solution 2.1

(a) The inverse of $t_{5,-3}$ is $t_{-5,3}$.

(b) The composite $t_{2,-3} \circ t_{4,5}$ is $t_{2+4,-3+5} = t_{6,2}$.

Solution 2.2

(a) The inverse of $r_{2\pi/3}$ is $r_{-2\pi/3}$. (The angle $-2\pi/3$ lies in the interval $(-\pi, \pi]$.)

(b) The composite $r_{3\pi/5} \circ r_{4\pi/5}$ is $r_{7\pi/5} = r_{-3\pi/5}$. (The angle $7\pi/5 - 2\pi = -3\pi/5$ lies in the interval $(-\pi, \pi]$.)

Solution 2.3

(a) The required translation is

$$\begin{aligned} t_{-3,-2} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x-3, y-2). \end{aligned}$$

(b) The required rotation is obtained by substituting $\theta = -\pi/3$ into

$$\begin{aligned} r_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta). \end{aligned}$$

Since

$$\cos\left(-\tfrac{1}{3}\pi\right) = \tfrac{1}{2} \quad \text{and} \quad \sin\left(-\tfrac{1}{3}\pi\right) = -\tfrac{1}{2}\sqrt{3},$$

the required rotation is

$$\begin{aligned} r_{-\pi/3} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \left(\frac{x + \sqrt{3}y}{2}, \frac{-\sqrt{3}x + y}{2}\right). \end{aligned}$$

(c) Since $\sqrt{3} = \tan(\pi/3)$, the line $y = \sqrt{3}x$ makes an angle $\pi/3$ with the positive x-axis, so the required reflection is obtained by substituting $\theta = \pi/3$ into

$$\begin{aligned} q_\theta : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto (x\cos(2\theta) + y\sin(2\theta), \\ &\qquad x\sin(2\theta) - y\cos(2\theta)). \end{aligned}$$

Since

$$\cos\left(\tfrac{2}{3}\pi\right) = -\tfrac{1}{2} \quad \text{and} \quad \sin\left(\tfrac{2}{3}\pi\right) = \tfrac{1}{2}\sqrt{3},$$

the required reflection is

$$\begin{aligned} q_{\pi/3} : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x, y) &\longmapsto \left(\frac{-x + \sqrt{3}y}{2}, \frac{\sqrt{3}x + y}{2}\right). \end{aligned}$$

Solution 2.4

For $g \circ f$ we have

$$g(f(x,y)) = g(y,x) = (y+2, x+2),$$

so the required function is

$$\begin{aligned} g \circ f : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) &\longmapsto (y+2, x+2). \end{aligned}$$

As you saw in Activity 2.9(b), f is a reflection in the line $y = x$; also g is a translation parallel to $y = x$ (two units up and two units right). Hence $g \circ f$ is a glide-reflection in the line $y = x$.

Solution 3.1

We have

$$\begin{aligned} \sin\left(\tfrac{11}{12}\pi\right) &= \sin\left(\tfrac{3}{4}\pi + \tfrac{1}{6}\pi\right) \\ &= \sin\left(\tfrac{3}{4}\pi\right)\cos\left(\tfrac{1}{6}\pi\right) + \cos\left(\tfrac{3}{4}\pi\right)\sin\left(\tfrac{1}{6}\pi\right) \\ &= \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right) \\ &= \tfrac{1}{4}\left(\sqrt{6} - \sqrt{2}\right). \end{aligned}$$

Also

$$\begin{aligned} \cos\left(\tfrac{11}{12}\pi\right) &= \cos\left(\tfrac{3}{4}\pi + \tfrac{1}{6}\pi\right) \\ &= \cos\left(\tfrac{3}{4}\pi\right)\cos\left(\tfrac{1}{6}\pi\right) - \sin\left(\tfrac{3}{4}\pi\right)\sin\left(\tfrac{1}{6}\pi\right) \\ &= \left(-\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) - \left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\right) \\ &= -\tfrac{1}{4}\left(\sqrt{6} + \sqrt{2}\right), \end{aligned}$$

and

$$\begin{aligned} \tan\left(\tfrac{11}{12}\pi\right) &= \tan\left(\tfrac{3}{4}\pi + \tfrac{1}{6}\pi\right) \\ &= \frac{\tan(\frac{3}{4}\pi) + \tan(\frac{1}{6}\pi)}{1 - \tan(\frac{3}{4}\pi)\tan(\frac{1}{6}\pi)} \\ &= \frac{-1 + 1/\sqrt{3}}{1 - (-1)/\sqrt{3}} \\ &= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} = -2 + \sqrt{3}, \end{aligned}$$

after rationalising the denominator.

Solution 3.2

Since θ lies in $\left(0, \tfrac{1}{2}\pi\right)$, it follows that $\cos\theta$ and $\sin\theta$, are positive, so

$$\cos\theta = \sqrt{\frac{1 + \cos(2\theta)}{2}} = \sqrt{\frac{1 - \frac{1}{9}}{2}} = \sqrt{\tfrac{4}{9}} = \tfrac{2}{3},$$

and

$$\sin\theta = \sqrt{\frac{1 - \cos(2\theta)}{2}} = \sqrt{\frac{1 + \frac{1}{9}}{2}} = \sqrt{\tfrac{5}{9}} = \tfrac{1}{3}\sqrt{5}.$$

Solution 3.3

Since $\cot\theta = -\frac{12}{5}$, we have $\tan\theta = -\frac{5}{12}$ and it follows by equation (3.3) that

$$\cos\theta = \pm\frac{1}{\sqrt{1 + (-\frac{5}{12})^2}} = \pm\tfrac{12}{13}.$$

Since $\frac{1}{2}\pi < \theta < \pi$, we have $\cos\theta = -\frac{12}{13}$.

Solution 4.1

(a) The inclination θ of L satisfies

$$\tan(2\theta) = \frac{B}{A - C} = \frac{4}{8 - 5} = \tfrac{4}{3},$$

so $\theta = \frac{1}{2}\arctan(4/3) \simeq 0.464$ radians (approximately 27°). Since $\tan(2\theta) = \frac{4}{3}$ a 3-4-5 triangle, gives

$$\cos(2\theta) = \tfrac{3}{5} \quad\text{and}\quad \sin(2\theta) = \tfrac{4}{5}.$$

By the half-angle formulas,

$$\begin{aligned} \cos^2\theta &= \tfrac{1}{2}(1 + \tfrac{3}{5}) = \tfrac{4}{5}, \\ \sin^2\theta &= \tfrac{1}{2}(1 - \tfrac{3}{5}) = \tfrac{1}{5}, \end{aligned}$$

and

$$\sin\theta\cos\theta = \tfrac{1}{2} \times \tfrac{4}{5} = \tfrac{2}{5}.$$

Hence, by equations (4.6),

$$\begin{aligned} A' &= 8 \times \tfrac{4}{5} + 4 \times \tfrac{2}{5} + 5 \times \tfrac{1}{5} = 9, \\ C' &= 8 \times \tfrac{1}{5} - 4 \times \tfrac{2}{5} + 5 \times \tfrac{4}{5} = 4. \end{aligned}$$

Since $F' = F = -36$, the equation of K is

$$9x^2 + 4y^2 - 36 = 0; \quad \text{that is,} \quad \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Hence K is an ellipse in reflected standard position with $a = 2$ and $b = 3$, and L is obtained by rotating K anticlockwise by approximately 27°. (Note that 27° is a little less than one-third of a right angle.)

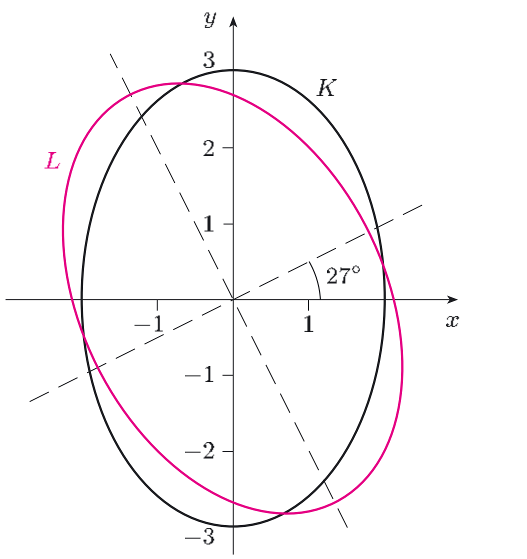


Figure S.14

(b) The inclination θ of L is $\frac{1}{4}\pi$, since $A = C = 1$, so

$$\cos^2\theta = \sin^2\theta = \sin\theta\cos\theta = \tfrac{1}{2}.$$

Hence, by equations (4.6),

$$A' = 1 \times \tfrac{1}{2} - 10 \times \tfrac{1}{2} + 1 \times \tfrac{1}{2} = -4,$$
$$C' = 1 \times \tfrac{1}{2} + 10 \times \tfrac{1}{2} + 1 \times \tfrac{1}{2} = 6.$$

Since $F' = F = 8$, the equation of K is

$$-4x^2 + 6y^2 + 8 = 0; \quad \text{that is,} \quad \tfrac{1}{2}x^2 - \tfrac{3}{4}y^2 = 1.$$

Hence K is a hyperbola in standard position with $a = \sqrt{2}$ and $b = 2/\sqrt{3}$, and L is obtained by rotating K anticlockwise by $\pi/4$ radians.

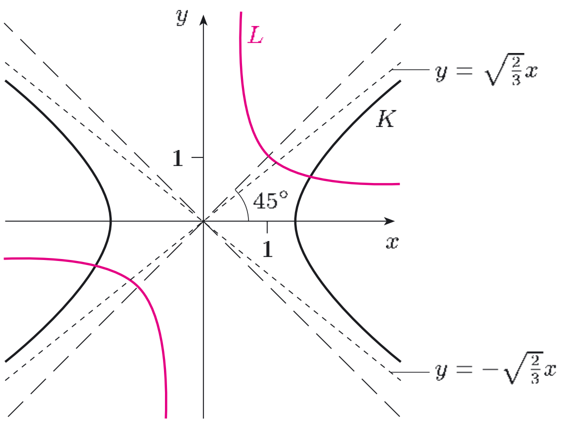


Figure S.15

Here K and L are shown on the same figure, but the asymptotes of L are not included.

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